Variable Seperable DE

Separable Equations (or Variables Separable Type): Here, the given differential equation can be reduced to the form f(y)dy = g(x)dx. [Recall that $\frac{dy}{dx}$ may be thought as the ratio of the differential of y to the differential of x]. Direct integration of the relation with respect to the variable on each side gives general solution or, in other words, the general solution of the differential equation above may be written as $\int f(y)\,dy = \int g(x)\,dx + C$, where C is an arbitrary constant.

Eg 1. Solve: $\frac{dy}{dx} = \sqrt{\frac{1+y^2}{1+x^2}}$

Sol. $\frac{dy}{dx} = \sqrt{\frac{1+y^2}{1+x^2}}$

$$\frac{1}{\sqrt{1+y^2}}dy = \frac{1}{\sqrt{1+x^2}}dx$$

Integrating on both sides,

$$\int\frac{1}{\sqrt{1+y^2}}\,dy = \int\frac{1}{\sqrt{1+x^2}}dx.$$

$\sinh^{-1}y = \sinh^{-1}x + c$

Eg 2. Solve: $(x - xy^2)\frac{dy}{dx} + (y + x^2y) = 0$

Sol. $(x - xy^2)\frac{dy}{dx} + (y + x^2y) = 0$

$(x - xy^2)\,dy + (y + x^2y)\,dx = 0$

$x(1 - y^2)\,dy + y(1 + x^2)\,dx = 0$

$$\frac{1-y^2}{y}dy + \frac{1+x^2}{x}dx = 0$$

Integrating on both sides,

$$\int\left(\frac{1}{y} - y\right)dy + \int\left(\frac{1}{x} + x\right)dx = 0$$

$$\log y - \frac{y^2}{2} + \log x + \frac{x^2}{2} = \log C$$

$$\log_e \frac{xy}{C} = \frac{y^2 - x^2}{2} \quad \frac{xy}{c} = e^{\left(\frac{y^2 - x^2}{2}\right)}$$

$$\Rightarrow xy = c\ e^{\left(\frac{y^2 - x^2}{2}\right)}$$

Eg 3. Solve the initial value problem $y^2 \frac{dy}{dx} = x^2 e^{y^3} . y(1) = 0$

Sol. Given: $y^2 \dfrac{dy}{dx} = x^2 e^{y^3}$

$$y^2\ e^{-y^3}\ dy = x^2 dx.$$

$$\int y^2\ e^{-y^3} dy = \int x^2 dx$$

Let $e^{-y^3} = t \Rightarrow e^{-y^3} . -3y^2 dy = dt$

$$-\frac{1}{3} \int dt = \int x^2 dx$$

$$\frac{-1}{3} t = \frac{x^3}{3} + c$$

$$-\frac{1}{3} e^{-y^3} = \frac{x^3}{3} + c\ .$$

Given: When x = 1, y = 0;

$$-\frac{1}{3} e^{\circ} = \frac{1}{3} + c$$

$$c = -\frac{2}{3}\ .$$

$\therefore$ The solution is $-\dfrac{1}{3} e^{-y^3} = \dfrac{x^3}{3} - \dfrac{2}{3}$.

$$x^3 + e^{-y3} - 2 = 0.$$