

## Properties of LTI Systems

### Properties of Continuous Time LTI Systems

**Systems with or without memory:** A system is memory less if its output at any time depends only on the value of the input at that same time.

$$\text{output } y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau$$

If  $y(t)$  need to be dependent only present input  $x(t)$  then  $h(\tau) x(t - \tau) = kx(t)$ , this is possible when  $h(\tau)$  is defined for only  $\tau = 0$

i.e.,  $h(t) = 0$  for  $t \neq 0$ .

The corresponding Impulse response  $h(t)$  of memory less system is simply  $h(t) = k\delta(t)$ . Therefore, if  $h(t_0) \neq 0$  for  $t_0 \neq 0$ , the continuous time LTI system has memory.

**Causality:** The output of a causal system depends only on the present and past values of input to the system.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau$$

$y(t)$  should not depend on future values, i.e.,  $h(t) = 0$  for  $t < 0$

For a causal continuous time LTI system, we have  $h(t) = 0$ ,  $t < 0$ .

The output of causal LTI system is

$$y(t) = \int_0^t h(\tau) x(t - \tau) d\tau = \int_0^t x(\tau) h(t - \tau) d\tau$$

**Stability:** A system is bounded input – bounded output (BIBO) stable if the output is guaranteed to be bounded for every bounded input.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau$$

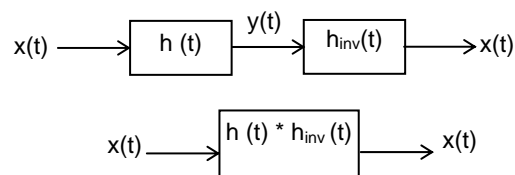
$|y(t)| < \infty$ , for  $|x(t)| < \infty$  then

$$|y(t)| = \int_{-\infty}^{\infty} |h(\tau) x(t - \tau)| d\tau = \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \text{ as } |x(t)| < \infty$$

A continuous time LTI system is BIBO stable if its impulse response is absolutely Integrable.

$$\text{i.e. } \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

**Invertibility:** If an LTI system is invertible, then it has an LTI inverse system, when the inverse system is connected in series with original system, it produces an output equal to the input to the first system.



$$\text{i.e., } x(t) = x(t) * \{h(t) * h_{\text{inv}}(t)\}$$

this will be valued only when

$$h(t) * h_{\text{inv}}(t) = \delta(t)$$

## Correlation

The correlation of function is a measure of similarity, regularity or coherence between two signals.

**Cross Correlation:** Cross correlation is the measure of similarity or regularity between two different signals. It is obtained between two signals by integrating the product of one signal and time delayed version of another signal.

Cross – correlation of the Energy (non periodic) signals  $x(t)$  and  $y(t)$

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)y(t - \tau)dt = \int_{-\infty}^{+\infty} y(t)x(t - \tau)dt$$

$$= \int_{-\infty}^{\infty} x(t)y(t + \tau)dt = \int_{-\infty}^{\infty} y(t)x(t + \tau)dt$$

For power signals (i.e., periodic signals with period  $\tau$ ) cross correlation is defined as

$$R_{xy}(\tau) = \frac{1}{T} \int_{-T/2}^{+T/2} x(t)y(t-\tau)dt = \frac{1}{T} \int_{-T/2}^{+T/2} x(t-\tau)y(t)dt$$

### Properties of Cross Correlation

- (1) Conjugate symmetry:  $R_{xy}(\tau) = R_{yx}^*(-\tau)$  [Correlation is not commutative]
- (2) If  $R_{xy}(0) = 0$ , then the two signals are called as orthogonal over the entire time interval.
- (3)  $R_{xy}(\tau) = R_{xy}^*(-\tau)$ , for complex valued signals.
- (4) The relation between cross correlation and convolution is  $R_{xy}(\tau) = x(\tau) * y^*(-\tau)$

**Auto Correlation:** It is a special form of cross correlation function. It is defined as the correlation of a signal with itself.

Auto correlation of energy signal  $x(t)$  is defined as  $R_{xx}(\tau) = \int_{-\infty}^{+\infty} x(t)x(t-\tau)dt$

$$= \int_{-\infty}^{+\infty} x(t+\tau)x(t)dt$$

Auto correlation of power signal (periodic with period  $T$ ) is defined as

$$R(\tau) = \frac{1}{T} \int_{-T/2}^{+T/2} x(t)x(t-\tau)dt = \frac{1}{T} \int_{-T/2}^{+T/2} x(t+\tau)x(t)dt$$

### Properties of Auto Correlation

- (1)  $R_{xx}(\tau) = R_{xx}(-\tau)$  for real valued signals
- (2)  $R_{xx}(\tau) = R_{xx}^*(-\tau)$  for complex valued signals

- (3) The Auto correlation function  $R_{xx}(\tau)$  at  $\tau = 0$  is equal to the total energy of the signal (or) average power of the signal.
- (4) If  $\tau$  is increased in either directions the auto correlation function  $R(\tau)$  decreases.  $R(\tau)$  is maximum at  $\tau = 0$   $|R(\tau)| \leq R(0)$  for all  $\tau$
- (5) The Auto correlation function and Energy spectral density function (or) power spectral density function forms a fourier transform pair  

$$R(\tau) \xleftrightarrow{FT} \psi(\omega) \text{ or } S(\omega)$$
- (6) For power signals (Periodic with fundamental period  $T$ ) the auto correlation function is also periodic with period  $T$ .  $R(\tau) = R(\tau + T)$

## Discrete Time LTI systems

1. Impulse Response  $h[n] = T\{\delta[n]\}$
2. Response to an Arbitrary input  $x[n]$  is  $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$
3. Convolution  $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$
4. Properties of convolution sum  
 $x[n] * h[n] = h[n] * x[n]$  – Commutative  
 $\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$   
– Associative  
 $x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$  – Distributive
5. Shifting property  $x_1[n] * x_2[n] = c[n]$   
Then  $x_1[n - n_1] * x_2[n - n_2] = c[n - n_1 - n_2]$
6. Convolution with impulse  $x[n] * \delta[n] = x[n]$ ,  
 $x[n - n_1] * \delta[n - n_2] = x[n - n_1 - n_2]$

7. The width property; if  $x_1[n]$  and  $x_2[n]$  have finite widths of  $w_1$  and  $w_2$ , respectively then the width of  $x_1[n] * x_2[n]$  is  $w_1 + w_2$ . The width of a signal is one less than the number of elements.

8. Causal system output; if  $x[n]$  and  $h[n]$  are causal, then  $y[n] = \sum_{k=0}^{\infty} x[k] h[n-k]$

9. Convolution sum table:

S.No.	$x_1[n]$	$x_2[n]$	$x_1[n] * x_2[n]$
1.	$\delta[n-k]$	$x[n]$	$x[n-k]$
2.	$u[n]$	$u[n]$	$(n+1) \cdot u[n]$
3.	$a^n u[n]$	$b^n u[n]$	$\left[ \frac{a^{n+1} - b^{n+1}}{a - b} \right] u[n], a \neq b$
4.	$u[n]$	$nu[n]$	$\frac{n(n+1)}{2} u[n]$
5.	$a^n u[n]$	$a^n u[n]$	$(n+1)a^n u[n]$

10. When two LTI Discrete time systems with impulse response  $h_1[n]$  and  $h_2[n]$  respectively are connected in parallel, the composite parallel system impulse response is  $h_1[n] + h_2[n]$ . If these systems are connected in cascade (in any order), the impulse response of the composite system is  $h_1[n] * h_2[n] = h_2[n] * h_1[n]$ .

11. Step response

$$s[n] = h[n] * u[n] = \sum_{k=0}^{\infty} h[n-k] = \sum_{k=-\infty}^n h[k]$$

$$h[n] = s[n] - s[n-1]$$

12. If  $h[n] \neq 0$ , for  $n \neq 0$ , the discrete time LTI system has memory (or)  $h[n] = 0$ , for  $n \neq 0$  system is memory less.

13. Causality condition is  $h[n] = 0$  for  $n < 0$ , causal system output

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} x[k]h[n-k]$$

14. A DT, LTI system is BIBO stable if  $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ .

15. Invertibility:  $h[n] * h_{inv}[n] = \delta[n]$

16. The steps involved in calculation of convolution

Continuous time convolution		Discrete – time convolution	
	$y(t) = x(t) * h(t) = h(t) * x(t)$ $= \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$		$y[n] = x[n] * h[n] = h[n] * x[n]$ $= \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$
1.	$x(t) \rightarrow x(\tau)$ (or) $h(t) \rightarrow h(\tau)$	1.	$h[n] \rightarrow h[k]$ , (or) $x[n] \rightarrow x[k]$
2.	Folding (or) flipping $x(-\tau)$ or $h(-\tau)$	2.	Folding (or) flipping $x[-k]$ or $h[-k]$
3.	Shifting $x(t-\tau)$ or $h(t-\tau)$	3.	Shifting $x[n-k]$ (or) $h[n-k]$
4.	Multiplication $x[t-\tau] h[\tau]$ (or) $h[t-\tau] x[\tau]$	4.	Multiplication $h[k] x[n-k]$ (or) $x[k] h[n-k]$
5.	Integration	5.	Summation

17. For Discrete time signals, correlation and Auto correlation are defined in terms of summation.

Cross correlation of two Energy Signals  $x[n]$  and  $y[n]$  is defined as  $R_{xy}([k])$

$$= \sum_{-\infty}^{+\infty} x[n]y[n-k] = \sum_{-\infty}^{+\infty} x[n+k]y[n]$$

Cross correlation of two power signals  $x[n]$  and  $y[n]$  is

$$R_{xy}[k] = N \xrightarrow{Lt} \frac{1}{2N+1} \sum_{n=-N}^N x[n]y[n-k]$$

If  $x[n]$  and  $y[n]$  are periodic discrete time sequences each with period  $N$  then cross correlation is defined as

$$R_{xy}[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]y[n-k]$$

18. Auto correlation of a sequence is correlation of a sequence with it self.

Auto correlation of discrete time energy signals  $x[n]$  is

$$R_{xx}[k] = \sum_{n=-\infty}^{\infty} x[n]x[n-k] = \sum_{n=-\infty}^{\infty} x[n+k]x[n]$$

Auto correlation of discrete – time power signals  $x[n]$  is defined as

$$R_{xx}[k] = N \xrightarrow{Lt} \infty \frac{1}{2N+1} \sum_{n=-N}^N x[n]x[n-k]$$

If  $x[n]$  is a periodic sequence with period  $N$ , then Auto correlation

$$R_{xx}[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]x[n-k]$$

19. Properties of correlation

(1) Conjugate symmetry:

$$R_{xy}[k] = R_{yx}^*[-k] \text{ if } x[n], y[n] \text{ are complex}$$

$$R_{xy}[k] = R_{xy}[-k] \text{ if } x[n], y[n] \text{ are real}$$

$$(2) \quad |R_{xy}[k]| \leq \sqrt{R_{xx}[0] \cdot R_{yy}[0]}$$

$$(3) \quad |R_{xx}[k]| \leq R_{xx}[0]$$

(4) The relation ship between convolution and correlation is  $R_{xy}[n] = x[n] * y[-n]$

20. The output  $y[n]$  of an LTI system, having impulse response  $h[n]$  and input  $x[n]$  is given by  $y[n] = x[n] * h[n]$

If  $h[n]$  and  $y[n]$  are known, then to find  $x[n]$

$$\text{We can use } x[n] = \frac{1}{h[0]} \left\{ y[n] - \sum_{k=0}^{n-1} x[k]h[n-k] \right\}$$

21. Circular or periodic convolution is defined as  $y[n] = x[n] \otimes h[n]$

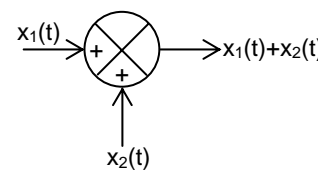
$$= \sum_{k=0}^{m-1} x[k]h[(n-k)_{\text{Mod } m}]$$

If  $x[n]$  and  $h[n]$  are of length  $m$  then the circular convolution of  $x[n]$  and  $h[n]$  is also of same length 'm'. if we need circular convolution of length 8. Then length of  $x[n]$  and  $h[n]$  should be 8. Hence we will pad appropriately number of zeros at end of  $x[n]$  and  $h[n]$  to adjust length, it is called as zero padding.

## Block Diagram Representation

Block diagram of a system is a pictorial representation of the function performed by the system

The basic elements of a block diagram of continuous time systems are given below.

Function	Block diagrams
Differentiator	$x(t) \rightarrow \left[ \frac{d}{dt} \text{ (or) } s \right] \rightarrow \frac{d}{dt} x(t)$
Integrator	$x(t) \rightarrow \left[ \int \text{ (or) } \frac{1}{s} \right] \rightarrow \int x(t) dt$
Constant multiplier	$x(t) \rightarrow a \rightarrow ax(t)$
Adder	

For discrete time systems, the difference equation can be represented in simple terms of block diagram. The basic elements of block diagram are given below.

Functions	Block diagrams
Unit delay element	$x[n] \rightarrow [z^{-1}] \rightarrow x[n - 1]$
Unit advance element	$x[n] \rightarrow [z] \rightarrow x[n + 1]$
Constant multiplier	$x[n] \rightarrow a \rightarrow ax[n]$
Adder	