

QUESTIONS & KEYS FOR JEE (ADVANCED)-2020 (PAPER 1)

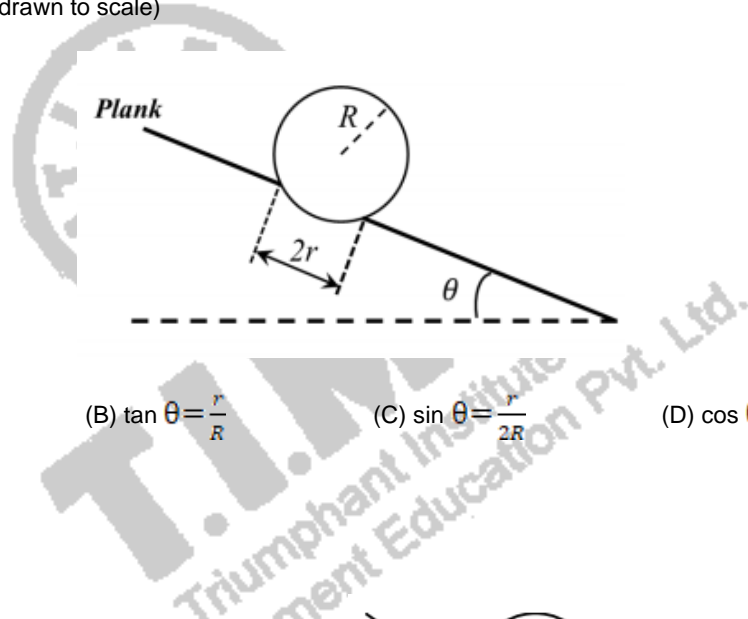
[PHYSICS, CHEMISTRY & MATHEMATICS]

[PART A - PHYSICS]

SECTION 1 (Maximum Marks 18)

- This section contains **SIX** (06) questions
- Each question **FOUR** option. **ONLY ONE** of these four option is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme.
Full Marks : +3 If **ONLY** the correct options is chosen;
Zero Marks : 0 If none of the options is chosen (i.e., the questions is unanswered);
Negative Marks : -1 In all other cases.

1. A football of radius R is kept on a hole of radius r ($r < R$) made on a plank kept horizontally. One end of the plank is now lifted so that it gets tilted making an angle θ from the horizontal as shown in the figure below. The maximum value of θ so that the football does not start rolling down the plank satisfies (figure is schematic and not drawn to scale)



(A) $\sin \theta = \frac{r}{R}$

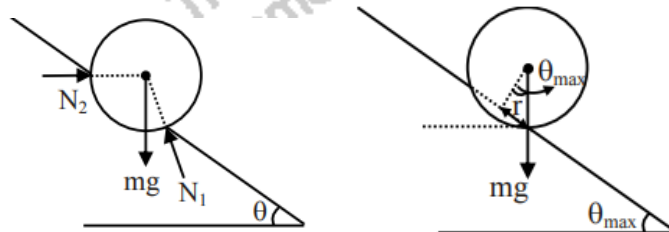
(B) $\tan \theta = \frac{r}{R}$

(C) $\sin \theta = \frac{r}{2R}$

(D) $\cos \theta = \frac{r}{2R}$

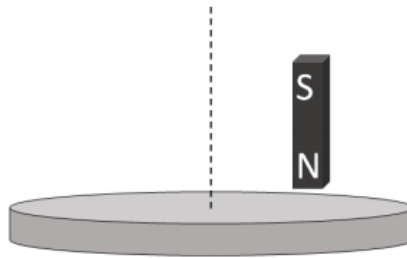
Answer-Key (A)

Sol:



For θ_{\max} , the football is about to roll, then $N_2 = 0$ and All the forces (Mg and N_1) must pass through contact point
 $\therefore \cos(90^\circ - \theta_{\max}) = \frac{r}{R} \Rightarrow \sin \theta_{\max} = \frac{r}{R}$

2. A light disc made of aluminium (a nonmagnetic material) is kept horizontally and is free to rotate about its axis as shown in the figure. A strong magnet is held vertically at a point above the disc away from its axis. On revolving the magnet about the axis of the disc, the disc will (figure is schematic and not drawn to scale)



- (A) rotate in the direction opposite to the direction of magnet's motion
 (B) rotate in the same direction as the direction of magnet's motion
 (C) not rotate and its temperature will remain unchanged
 (D) not rotate but its temperature will slowly rise

Answer Key (B)

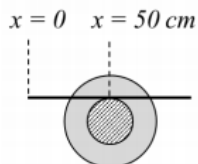
Sol:

When the magnet is moved, it creates a state where the plate moves through the magnetic flux, due to which an electromotive force is generated in the plate and eddy currents are induced. These eddy currents opposes the relative motion \Rightarrow disc will rotate in the direction of rotation of magnet.

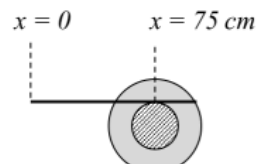
3. A small roller of diameter 20 cm has an axle of diameter 10 cm (see figure below on the left). It is on a horizontal floor and a meter scale is positioned horizontally on its axle with one edge of the scale on top of the axle (see figure on the right). The scale is now pushed slowly on the axle so that it moves without slipping on the axle, and the roller starts rolling without slipping. After the roller has moved 50 cm, the position of the scale will look like (figures are schematic and not drawn to scale)



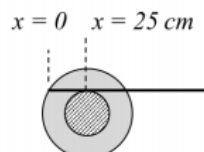
(A)



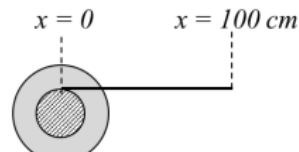
(B)



(C)



(D)



Answer Key (C)

Sol:

For no slipping at the ground, $V_{\text{centre}} = \omega R$ (R is radius of roller)

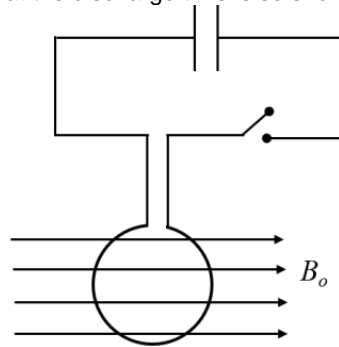
\therefore Velocity of scale = $(V_{\text{center}} + \omega r)$ [r is radius of axle] Given, $V_{\text{center}} \cdot t = 50 \text{ cm}$

$$\therefore \text{Distance moved by scale} = (V_{\text{center}} + \omega r)t$$

$$= \left(V_{\text{center}} + \frac{V_{\text{center}} r}{R} \right) t = \frac{3V_{\text{center}}}{2} t$$

$$t = 75 \text{ cm}$$

4. A circular coil of radius R and N turns has negligible resistance. As shown in the schematic figure, its two ends are connected to two wires and it is hanging by those wires with its plane being vertical. The wires are connected to a capacitor with charge Q through a switch. The coil is in a horizontal uniform magnetic field B_0 parallel to the plane of the coil. When the switch is closed, the capacitor gets discharged through the coil in a very short time. By the time the capacitor is discharged fully, magnitude of the angular momentum gained by the coil will be (assume that the discharge time is so short that the coil has hardly rotated during this time)



- (A) $\frac{\pi}{2} NQB_0R^2$
 (C) $2\pi NQB_0R^2$

- (B) πNQB_0R^2
 (D) $4\pi NQB_0R^2$

Answer Key (B)

Sol:

Torque experienced by circular loop = $\vec{M} \times \vec{B}$

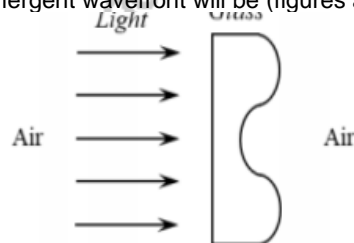
where \vec{M} is magnetic moment

\vec{B} is magnetic field

$$\therefore \tau = i\pi R^2 N B_0 \text{ [at the instant shown } \theta = \pi/2]$$

$$\therefore \tau dt = d\vec{L} = i\pi R^2 N B_0 dt = Q\pi R^2 N B_0 \text{ [} \int dt = Q]$$

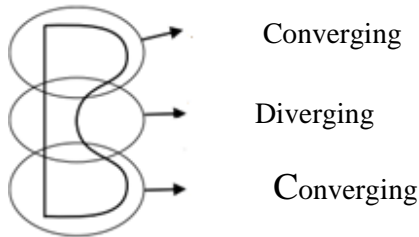
5. A parallel beam of light strikes a piece of transparent glass having cross section as shown in the figure below. Correct shape of the emergent wavefront will be (figures are schematic and not drawn to scale)



- (A) (B) (C) (D)

Answer Key (A)

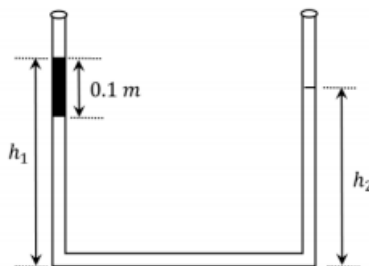
Sol:



(A)



6. An open-ended U-tube of uniform cross-sectional area contains water (density 10^3 kg m^{-3}). Initially the water level stands at 0.29 m from the bottom in each arm. Kerosene oil (a water-immiscible liquid) of density 800 kg m^{-3} is added to the left arm until its length is 0.1 m, as shown in the schematic figure below. The ratio $\left(\frac{h_1}{h_2}\right)$ of the heights of the liquid in the two arms is



(A) $\frac{15}{14}$

(B) $\frac{35}{33}$

(C) $\frac{7}{6}$

(D) $\frac{5}{4}$

Answer Key (B)

Sol:

$$h_1 + h_2 = 0.29 \times 2 + 0.1$$

$$h_1 + h_2 = 0.68 \dots\dots(1)$$

$$\Rightarrow P_0 + \rho_k g(0.1) + \rho_w g(h_1 - 0.1) \quad [\rho_k = \text{density of kerosene} \ \& \ \rho_w = \text{density of water}]$$

$$-\rho_k g h_2 = P_0$$

$$\Rightarrow \rho_k g(0.1) + \rho_w g h_1 - \rho_w g \times (0.1)$$

$$= \rho_w g h_2$$

$$\Rightarrow 800 \times 10 \times 0.1 + 1000 \times 10 \times h_1$$

$$- 1000 \times 10 \times 0.1 = 1000 \times 10 \times h_2$$

$$\Rightarrow 10000 (h_1 - h_2) = 200$$

$$\Rightarrow h_1 - h_2 = 0.02 \dots\dots(2)$$

$$\Rightarrow h_1 = 0.35$$

$$\Rightarrow h_2 = 0.33$$

$$\text{So, } \frac{h_1}{h_2} = \frac{35}{33}$$

SECTION 2 (Maximum Marks 24)

- This section contains **SIX** (06) questions
- Each question **FOUR** option. **ONLY ONE OR MORE THAN** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks	:	+4	If only (all) the correct option(s) is (are) chosen:
Partial Marks	:	+3	If all the four options are correct but ONLY three options are chosen;
Partial Marks	:	+2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
Partial Marks	:	+1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
Zero Marks	:	0	If none of the options is chosen (i.e., the questions is unanswered);
Negative Marks	:	-2	In all other cases.

7. A particle of mass m moves in circular orbits with potential energy $V(r) = Fr$, where F is a positive constant and r is its distance from the origin. Its energies are calculated using the Bohr model. If the radius of the particle's orbit is denoted by R and its speed and energy are denoted by v and E , respectively, then for the n^{th} orbit (here h is the Planck's constant)

- (A) $R \propto n^{1/3}$ and $v \propto n^{2/3}$ (B) $R \propto n^{2/3}$ and $v \propto n^{1/3}$
 (C) $E = \frac{3}{2} \left(\frac{n^2 h^2 F^2}{4\pi^2 m} \right)^{1/3}$ (D) $E = 2 \left(\frac{n^2 h^2 F^2}{4\pi^2 m} \right)^{1/3}$

Answer Key (B, C)

Sol:

$U = Fr$

[Using $U =$ Potential energy and $v =$ velocity, to avoid confusion between their symbols]

\Rightarrow Force = $-\frac{dU}{dr} = -F$

\Rightarrow Magnitude of force = Constant = F

$\Rightarrow F = \frac{mv^2}{R}$ (1)

$\Rightarrow mvR = \frac{nh}{2\pi}$ (2)

$\Rightarrow F = \frac{m}{R} \times \frac{n^2 h^2}{4\pi^2} \times \frac{1}{m^2 R^2}$

$\Rightarrow R = \left(\frac{n^2 h^2}{4\pi^2 m F} \right)^{1/3}$ (3)

$\Rightarrow v = \frac{nh}{2\pi m R}$

$\Rightarrow v = \frac{nh}{2\pi m \left(\frac{4\pi^2 m F}{n^2 h^2} \right)^{1/3}}$
 $\Rightarrow v = \frac{n^{1/3} h^{1/3} F^{2/3}}{2^{1/3} \pi^{1/3} m^{2/3}}$ (4)

(B) is correct

$\Rightarrow E = \frac{1}{2}mv^2 + U = \frac{1}{2}mv^2 + FR$

$\Rightarrow E = \frac{1}{2}m \left(\frac{n^{1/3} h^{1/3} F^{2/3}}{2^{1/3} \pi^{1/3} m^{2/3}} \right)^2 + F \times \left(\frac{n^2 h^2}{4\pi^2 m F} \right)^{1/3}$

$\Rightarrow E = \left(\frac{n^2 h^2 F^2}{4\pi^2 m} \right)^{1/3} \left[\frac{1}{2} + 1 \right] = \frac{3}{2} \left(\frac{n^2 h^2 F^2}{4\pi^2 m} \right)^{1/3}$

8. The filament of a light bulb has surface area 64 mm^2 . The filament can be considered as a black body at temperature 2500 K emitting radiation like a point source when viewed from far. At night the light bulb is observed from a distance of 100 m . Assume the pupil of the eyes of the observer to be circular with radius 3 mm . Then (Take Stefan-Boltzmann constant = $5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$, Wien's displacement constant = $2.90 \times 10^{-3} \text{ m-K}$, Planck's constant = $6.63 \times 10^{-34} \text{ Js}$, speed of light in vacuum = $3.00 \times 10^8 \text{ ms}^{-1}$)

- (A) power radiated by the filament is in the range 642 W to 645 W
 (B) radiated power entering into one eye of the observer is in the range $3.15 \times 10^{-8} \text{ W}$ to $3.25 \times 10^{-8} \text{ W}$
 (C) the wavelength corresponding to the maximum intensity of light is 1160 nm
 (D) taking the average wavelength of emitted radiation to be 1740 nm , the total number of photons entering per second into one eye of the observer is in the range 2.75×10^{11} to 2.85×10^{11}

Answer Key (B, C, D)

Sol:

$A = 64 \text{ mm}^2$, $T = 2500 \text{ K}$ (A = surface area of filament, T = temperature of filament, d is distance of bulb from observer, R_e = radius of pupil of eye)

Point source $d = 100 \text{ m}$

$R_e = 3 \text{ mm}$

(A) $P = \sigma A e T^4 = 5.67 \times 10^{-8} \times 64 \times 10^{-6} \times 1 \times (2500)^4$ ($e = 1$ black body) = 141.75 w Option (A) is wrong

(B) Power reaching to the eye

$$= \frac{P}{4\pi d^2} \times (\pi R_e^2)$$

$$= \frac{141.75}{4\pi \times (100)^2} \times \pi \times (3 \times 10^{-3})^2$$

$$= 3.189375 \times 10^{-8} \text{ W}$$

Option (B) is correct

(C) $\lambda_m T = b$

$$\lambda_m \times 2500 = 2.9 \times 10^{-3}$$

$$\Rightarrow \lambda_m = 1.16 \times 10^{-6} = 1160 \text{ nm}$$
 Option (C) is correct

(D) Power received by one eye of observer = $\left(\frac{hc}{\lambda}\right) \times \dot{N}$

$$\dot{N} = \text{Number of photons entering into eye per second} \Rightarrow 3.189375 \times 10^{-8} = \frac{6.63 \times 10^{-34} \times 3 \times 10^8}{1740 \times 10^{-9}} \times \dot{N}$$

$$\Rightarrow \dot{N} = 2.79 \times 10^{11}$$

Option (D) is correct

9. Sometimes it is convenient to construct a system of units so that all quantities can be expressed in terms of only one physical quantity. In one such system, dimensions of different quantities are given in terms of a quantity X as follows: [position] = $[X^\alpha]$; [speed] = $[X^\beta]$; [acceleration] = $[X^\gamma]$; [linear momentum] = $[X^q]$; [force] = $[X^r]$. Then

(A) $\alpha + p = 2\beta$

(B) $p + q - r = \beta$

(C) $p - q + r = \alpha$

(D) $p + q + r = \beta$

Answer Key (A, B)

Sol:

Given $L = x^\alpha$ (1)

$LT^{-1} = x^\beta$ (2)

$LT^{-2} = x^p$ (3)

$MLT^{-1} = x^q$ (4)

$MLT^{-2} = x^r$ (5)

$\frac{(1)}{(2)} \Rightarrow T = x^{\alpha-\beta}$

(2)

From (3)

$$\frac{x^\alpha}{x^{2(\alpha-\beta)}} = x^p$$

$$\Rightarrow \alpha + p = 2\beta$$
 (A)

From (4)

$$M = x^{q-\beta}$$

From (5) $\Rightarrow x^q = x^r x^{\alpha-\beta}$

$$\Rightarrow \alpha + r - q = \beta$$
(6)

Replacing value ' α ' in equation (6) from (A)

$$2\beta - p + r - q = \beta$$

$$\Rightarrow p + q - r = \beta$$
 (B)

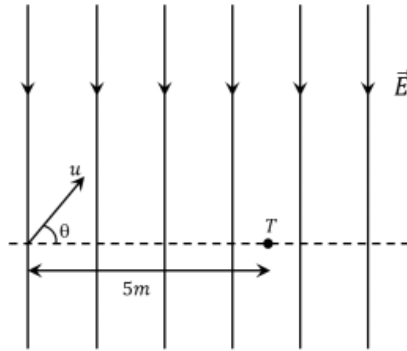
Replacing value of ' β ' in equation (6) from (A)

$$2\alpha + 2r - 2q = \alpha + p$$

$$\alpha = p + 2q - 2r$$

10. A uniform electric field, $E = -400\sqrt{3}\hat{y} \text{ NC}^{-1}$ is applied in a region. A charged particle of mass m carrying positive charge q is projected in this region with an initial speed of $2\sqrt{10} \times 10^6 \text{ ms}^{-1}$. This particle is aimed to hit a target T, which is 5 m away from its entry point into the field as shown schematically in the figure.

Take $\frac{q}{m} = 10^{10} \text{ Ckg}^{-1}$. Then



- (A) the particle will hit T if projected at an angle 45° from the horizontal
 (B) the particle will hit T if projected either at an angle 30° or 60° from the horizontal
 (C) time taken by the particle to hit T could be $\sqrt{\frac{5}{6}} \mu\text{s}$ as well as $\sqrt{\frac{5}{2}} \mu\text{s}$
 (D) time taken by the particle to hit T is $\sqrt{\frac{5}{3}} \mu\text{s}$

Answer Key (B, C)

Sol:

$$a_y = -400\sqrt{3} \times 10^{10} \text{ [} qE_y = ma_y \text{]}$$

$$R = 5 = \frac{40 \times 10^{12} \sin 2\theta}{400\sqrt{3} \times 10^{10}} \left[R(\text{range}) = \frac{u^2 \sin 2\theta}{a_y} \right]$$

$$\sin 2\theta = \frac{\sqrt{3}}{2}$$

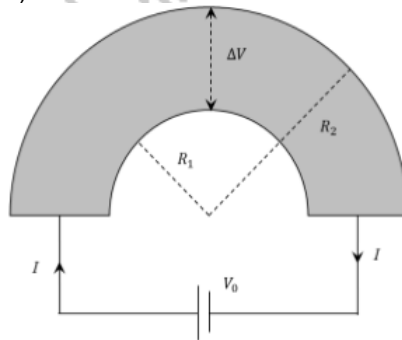
$$2\theta = 60^\circ, 120^\circ \Rightarrow \theta = 30^\circ, 60^\circ$$

$$\text{Time of flight } T_1 = \frac{2 \times 2\sqrt{10} \times 10^6 \times \frac{1}{2}}{400\sqrt{3} \times 10^{10}}$$

$$= \sqrt{\frac{5}{6}} \mu\text{s (for } \theta = 30^\circ \text{)}$$

$$\text{Time of flight } T_2 = \frac{2 \times 2\sqrt{10} \times 10^6 \times \frac{\sqrt{3}}{2}}{400\sqrt{3} \times 10^{10}} = \sqrt{\frac{5}{2}} \mu\text{s (for } \theta = 60^\circ \text{)}$$

11. Shown in the figure is a semicircular metallic strip that has thickness t and resistivity ρ . Its inner radius is R_1 and outer radius is R_2 . If a voltage V_0 is applied between its two ends, a current I flows in it. In addition, it is observed that a transverse voltage ΔV develops between its inner and outer surfaces due to purely kinetic effects of moving electrons (ignore any role of the magnetic field due to the current). Then (figure is schematic and not drawn to scale)

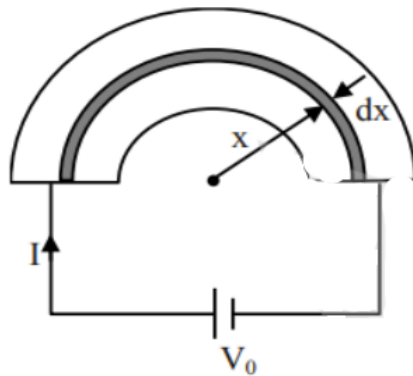


(A) $I = \frac{V_0 t}{\pi \rho} \ln \left(\frac{R_2}{R_1} \right)$

- (B) the outer surface is at a higher voltage than the inner surface
 (C) the outer surface is at a lower voltage than the inner surface
 (D) $\Delta V \propto I^2$

Answer Key (A, C, D)

Sol:



All the elements are in parallel

$$\therefore \int \frac{1}{dr} = \int_{R_1}^{R_2} \frac{tdx}{\rho \pi x}$$

$$\frac{l}{r} = \frac{t}{\rho \pi} \ln \left(\frac{R_2}{R_1} \right)$$

$$\text{Resistance} = \frac{\rho \pi}{t \ln \left(\frac{R_2}{R_1} \right)}$$

$$i = \frac{V_0 t \ln \left(\frac{R_2}{R_1} \right)}{\rho \pi} \dots\dots(A)$$

$(-e\vec{E})$ will be inward direction in order to provide centripetal acceleration. Therefore electric field will be radially outward

$$V_{\text{outer}} < V_{\text{inner}} \dots\dots(C)$$

$$\frac{mV_d^2}{r} = q\vec{E}$$

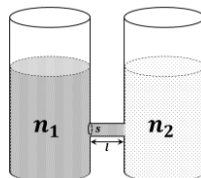
$$\vec{E} = \frac{mV_d^2}{qr} \quad (I = neAV_d \Rightarrow V_d \propto i)$$

$$\Delta V = \int \vec{E} \cdot d\vec{r}$$

$$\Delta V \propto V_d^2$$

$$\Delta V \propto I^2$$

12. As shown schematically in the figure, two vessels contain water solutions (at temperature T) of potassium permanganate (KMnO_4) of different concentrations n_1 and n_2 ($n_1 > n_2$) molecules per unit volume with $\Delta n = (n_1 - n_2) \ll n_1$. When they are connected by a tube of small length l and cross-sectional area S , KMnO_4 starts to diffuse from the left to the right vessel through the tube. Consider the collection of molecules to behave as dilute ideal gases and the difference in their partial pressure in the two vessels causing the diffusion. The speed v of the molecules is limited by the viscous force $-\beta v$ on each molecule, where β is a constant. Neglecting all terms of the order $(\Delta n)^2$, which of the following is/are correct? (k_B is the Boltzmann constant)



(A) the force causing the molecules to move across the tube is $\Delta n k_B T S$

(B) force balance implies $n_1 \beta v l = \Delta n k_B T$

(C) total number of molecules going across the tube per sec is $\left(\frac{\Delta n}{l} \right) \left(\frac{k_B T}{\beta} \right) S$

(D) rate of molecules getting transferred through the tube does not change with time

Answer Key (A, B, C)

Sol:

$$n \gg (n_1 - n_2) = \Delta n$$

$$p_1 = \frac{n_1 RT}{N_A} \quad p_2 = \frac{n_2 RT}{N_A}$$

$$F = (n_1 - n_2)k_B T S = \Delta n k_B T S \quad (A)$$

$$V = \frac{\Delta n k_B T S}{\beta}$$

Force balance \Rightarrow Pressure \times Area = Total number of molecules $\times \beta v$

$$\Delta n k_B T S = n_1 \beta v$$

$$\Rightarrow n_1 \beta v l = \Delta n k_B T \quad (B)$$

$$\text{Total number of molecules/sec} = \frac{(n_1 v dt) S}{dt} = n_1 V s = \frac{\Delta n k_B T v S}{\beta v l} = \left(\frac{\Delta n}{l}\right) \left(\frac{k_B T}{\beta}\right) S \quad \dots (C)$$

As Δn will decrease with time, rate of molecules getting transfer decreases with time.

SECTION 3 (Maximum Marks 24)

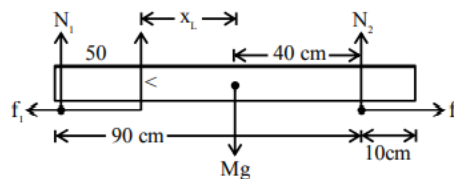
- This section contains **SIX** (06) questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate / round-off** the value of **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks	:	+4	If ONLY the correct numerical value is entered.
Zero Marks	:	0	In all other cases.

13. Put a uniform meter scale horizontally on your extended index fingers with the left one at 0.00 cm and the right one at 90.00 cm. When you attempt to move both the fingers slowly towards the center, initially only the left finger slips with respect to the scale and the right finger does not. After some distance, the left finger stops and the right one starts slipping. Then the right finger stops at a distance x_R from the center (50.00 cm) of the scale and the left one starts slipping again. This happens because of the difference in the frictional forces on the two fingers. If the coefficients of static and dynamic friction between the fingers and the scale are 0.40 and 0.32, respectively, the value of x_R (in cm) is _____.

Answer Key (25.60)

Sol:



Initially

$$N_1 + N_2 = Mg$$

$$(\tau_N = 0) N_1(50) = N_2(40)$$

About centre

$$5N_1 = 4N_2$$

$$f_{1k} = \mu_k N_1 \quad f_{1L} = \mu_s N_1$$

$$f_{1k} = 0.32N_1 \quad f_{1L} = 0.4N_1$$

$$f_{2k} = 0.32N_2 \quad f_{2L} = 0.4N_2$$

$$N_1 = \frac{4Mg}{9}$$

$$N_2 = \frac{5Mg}{9}$$

Suppose x_L = distance of left finger from centre when right finger starts moving

$$(\tau_N = 0)_{\text{about centre}} \Rightarrow N_1 x_L = N_2(40)$$

$$f_{k1} = f_{L2} \Rightarrow 0.32 N_1 = 0.40 N_2$$

$$4N_1 = 5N_2$$

$$N_1 x_L = \frac{4N_2}{5} (40)$$

$$x_L = 32$$

Now x_R = distance when right finger stops and left finger starts moving

$$(\tau_N = 0)_{\text{about centre}} \Rightarrow N_1 x_L = N_2(x_R)$$

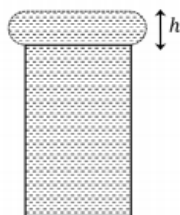
$$f_{L1} = f_{k2} \Rightarrow 0.4 N_1 = 0.32 N_2$$

$$5N_1 = 4N_2$$

$$\frac{4N_2}{5} (32) = N_2 x_R$$

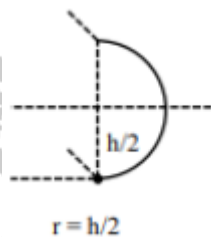
$$x_R = \frac{128}{5} = 25.6 \text{ cm}$$

14. When water is filled carefully in a glass, one can fill it to a height h above the rim of the glass due to the surface tension of water. To calculate h just before water starts flowing, model the shape of the water above the rim as a disc of thickness h having semicircular edges, as shown schematically in the figure. When the pressure of water at the bottom of this disc exceeds what can be withstood due to the surface tension, the water surface breaks near the rim and water starts flowing from there. If the density of water, its surface tension and the acceleration due to gravity are 10^3 kg m^{-3} , 0.07 Nm^{-1} and 10 ms^{-2} , respectively, the value of h (in mm) is _____.



Answer Key (3.74)

Sol:



Pressure at the bottom of disc = pressure due to surface tension

$$\rho gh = T \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

$$R_1 \gg R_2$$

$$\text{So } \frac{1}{R_1} \ll \frac{1}{R_2} \text{ and } R_2 = h/2$$

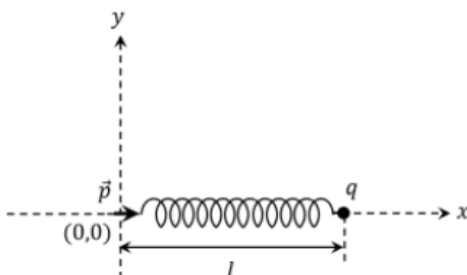
$$\therefore \rho gh = T \left[\frac{1}{R_1} + \frac{1}{R_2} \right] = T \left[0 + \frac{1}{h/2} \right]$$

$$h^2 = \frac{2T}{\rho g}$$

$$h = \sqrt{\frac{2T}{\rho g}} = \sqrt{\frac{2 \times 0.07}{10^3 \times 10}} = \sqrt{\frac{14 \times 100}{10^4 \times 100}}$$

$$h = \sqrt{14} \text{ mm} = 3.741$$

15. One end of a spring of negligible unstretched length and spring constant k is fixed at the origin $(0,0)$. A point particle of mass m carrying a positive charge q is attached at its other end. The entire system is kept on a smooth horizontal surface. When a point dipole \vec{p} pointing towards the charge q is fixed at the origin, the spring gets stretched to a length l and attains a new equilibrium position (see figure below). If the point mass is now displaced slightly by $\Delta l \ll l$ from its equilibrium position and released, it is found to oscillate at frequency $\frac{1}{\delta} \sqrt{\frac{k}{m}}$. The value of δ is _____



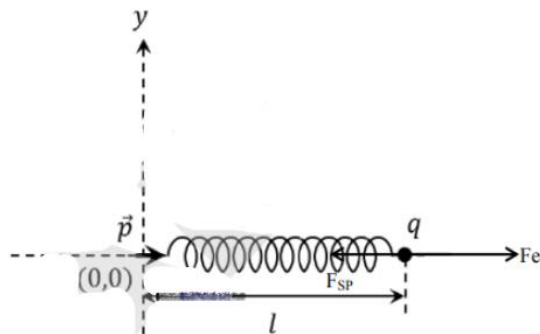
Answer Key (3.14)

Sol:

$$\Delta l \rightarrow x$$

$$\text{At } l: F_e = F_{sp}$$

$$kl = \frac{2kpq}{l^2}$$



$$F_{net} = F_{sp} - F_e = k(l+x) - \frac{q(2kp)}{(l+x)^2} = k(x+l) - \frac{q(2kp)}{l^2(1+x/l)^2} = kx + kl - q \left(\frac{2kp}{l^2} \right) \left(1 - \frac{3x}{l} \right) = kx + kl - q \left(\frac{2kp}{l^2} \right) + \frac{6kpq}{l^3} x$$

$$F_N = kx + kl \frac{3x}{l} = 4kx$$

$$K_{eq} = 4k$$

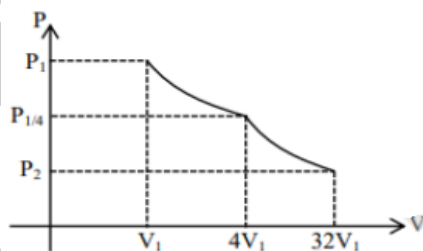
$$f = \frac{1}{\pi} \sqrt{\frac{k}{m}}$$

$$\text{So } \delta = \pi = 3.14$$

16. Consider one mole of helium gas enclosed in a container at initial pressure P_1 and volume V_1 . It expands isothermally to volume $4V_1$. After this, the gas expands adiabatically and its volume becomes $32V_1$. The work done by the gas during isothermal and adiabatic expansion processes are W_{iso} and W_{adia} , respectively. If the ratio $\frac{W_{iso}}{W_{adia}} = f \ln 2$, then f is _____

Answer Key (1.77 to 1.78)

Sol:



$$\frac{P_1}{4} (4V_1)^{5/3} = P_2 (32V_1)^{5/3}$$

$$P_2 = \frac{P_1}{4} \left(\frac{1}{8} \right)^{5/3} = \frac{P_1}{128}$$

$$W_{adi} = \frac{P_1 V_1 (3/4)}{2/3} = \frac{9}{8} P_1 V_1$$

$$W_{iso} = P_1 V_1 \ln \frac{4V_1}{V_1} = 2P_1 V_1 \ln 2$$

$$\frac{W_{iso}}{W_{adia}} = \frac{2P_1 V_1 \ln 2}{\frac{9}{8} P_1 V_1} = \frac{16}{9} \ln 2 = f \ln 2$$

$$f = \frac{16}{9} = 1.7778 \approx 1.78$$

17. A stationary tuning fork is in resonance with an air column in a pipe. If the tuning fork is moved with a speed of 2 ms^{-1} in front of the open end of the pipe and parallel to it, the length of the pipe should be changed for the resonance to occur with the moving tuning fork. If the speed of sound in air is 320 ms^{-1} , the smallest value of the percentage change required in the length of the pipe is _____.

Answer Key (0.62 to 0.63)

Sol:

$$f \propto \frac{1}{l_1} \Rightarrow f = \frac{k}{l_1} \dots\dots(1)$$

($l_1 \Rightarrow$ initial length of pipe)

$$\left(\frac{V}{V-V_T}\right) f = \frac{k}{l_2}$$

{V Speed of tuning fork, $l_2 \rightarrow$ new length of pipe} $\dots\dots(2)$

(1) \div (2)

$$\frac{V - V_T}{V} = \frac{l_2}{l_1}$$

$$\frac{l_2}{l_1} - 1 = \frac{V - V_T}{V} - 1$$

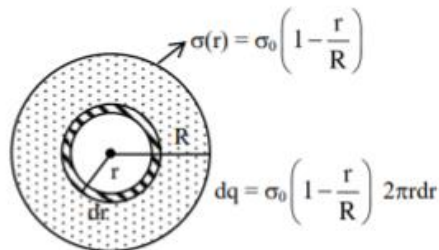
$$\frac{l_2 - l_1}{l_1} = \frac{-V_T}{V}$$

$$\frac{l_2 - l_1}{l_1} \times 100 = \frac{-2}{320} \times 100 = -0.625$$

18. A circular disc of radius R carries surface charge density $\sigma(r) = \sigma_0 \left(1 - \frac{r}{R}\right)$, where σ_0 is a constant and r is the distance from the center of the disc. Electric flux through a large spherical surface that encloses the charged disc completely is ϕ_0 . Electric flux through another spherical surface of radius $\frac{R}{4}$ and concentric with the disc is ϕ . Then the ratio $\frac{\phi_0}{\phi}$ is _____

Answer Key (6.40)

Sol:



$$\phi_0 = \frac{\int dq \cdot \epsilon_0 \sigma_0 \left(1 - \frac{r}{R}\right) 2\pi r dr}{\epsilon_0}$$

$$\phi = \frac{\int dq \cdot \epsilon_0 \sigma_0 \left(1 - \frac{r}{R}\right) 2\pi r dr}{\epsilon_0}$$

$$\therefore \frac{\phi_0}{\phi} = \frac{\int_0^R \sigma_0 2\pi \epsilon_0 \sigma_0 \left(1 - \frac{r}{R}\right) r dr}{\int_0^{R/4} \sigma_0 2\pi \epsilon_0 \left(1 - \frac{r}{R}\right) r dr}$$

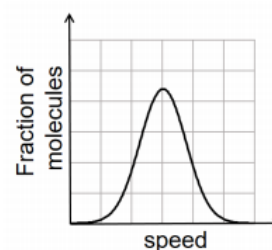
$$\frac{\frac{R^2}{2} - \frac{R^2}{4}}{\frac{R^2}{8} - \frac{R^2}{16}} = \frac{32}{5} = 6.40$$

[PART B - CHEMISTRY]

SECTION 1 (Maximum Marks 18)

- This section contains **SIX** (06) questions
 - Each question **FOUR** option. **ONLY ONE** of these four option is the correct answer.
 - For each question, choose the option corresponding to the correct answer.
 - Answer to each question will be evaluated according to the following marking scheme.
- Full Marks* : +3 If **ONLY** the correct options is chosen;
Zero Marks : 0 If none of the options is chosen (i.e., the questions is unanswered);
Negative Marks : -1 In all other cases.

1. If the distribution of molecular speeds of a gas is as per the figure shown below, then the ratio of the most probable, the average and the root mean square speeds, respectively, is



- (A) 1 : 1 : 1
(C) 1 : 1.128 : 1.224

- (B) 1 : 1 : 1.224
(D) 1 : 1.128 : 1

- (B) 1 : 1 : 1.224
(D) 1 : 1.128 : 1

Answer Key (B)

Sol:

From the graph it can be seen that most probable speed = Average speed
 But RMS speed is always greater than Av. Speed,
 \therefore MPV : AV : RMS = 1 : 1 : 1.224

2. Which of the following liberates O_2 upon hydrolysis?
 (A) Pb_3O_4 (B) KO_2 (C) Na_2O_2 (D) Li_2O_2

Answer Key (B)

Sol:

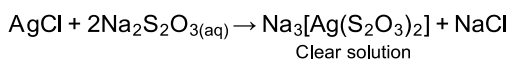
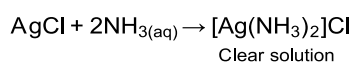
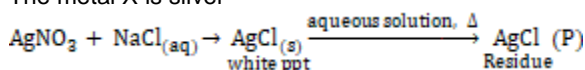
Superoxides undergo hydrolysis to liberate O_2
 $2KO_2 + 2H_2O \rightarrow 2KOH + H_2O_2 + O_2$

3. A colorless aqueous solution contains nitrates of two metals, **X** and **Y**. When it was added to an aqueous solution of $NaCl$, a white precipitate was formed. This precipitate was found to be partly soluble in hot water to give a residue **P** and a solution **Q**. The residue **P** was soluble in $aq.NH_3$ and also in excess sodium thiosulfate. The solution **Q** gave a yellow precipitate with KI . The metals **X** and **Y**, respectively, are
 (A) Ag and Pb (B) Ag and Cd (C) Cd and Pb (D) Cd and Zn

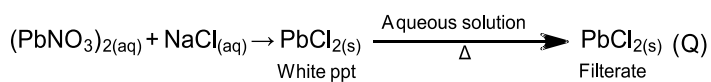
Answer Key (A)

Sol:

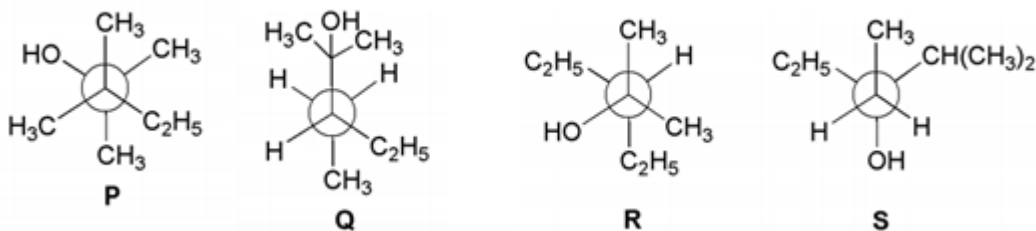
The metal X is silver



The metal Y is lead



4. Newman projections, **P**, **Q**, **R** and **S** are shown below:

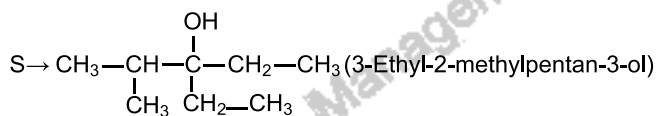
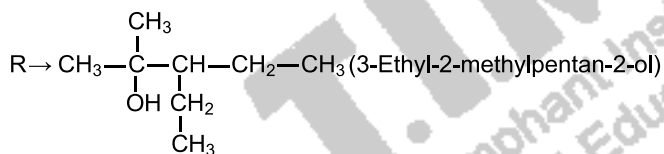
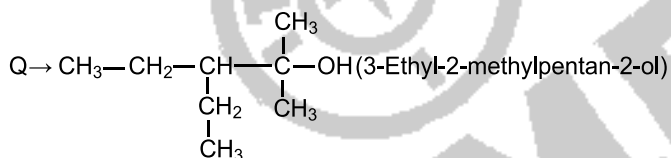
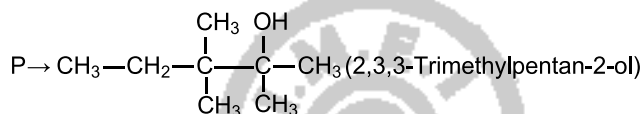


Which one of the following options represents identical molecules?

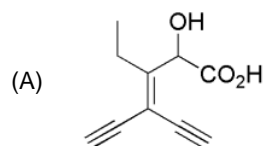
- (A) **P** and **Q** (B) **Q** and **S** (C) **Q** and **R** (D) **R** and **S**

Answer Key (C)

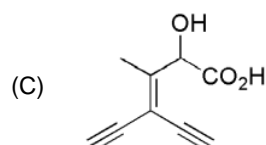
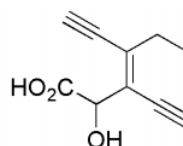
Sol:



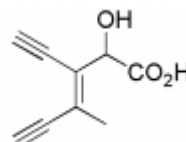
5. Which one of the following structures has IUPAC name 3-ethynyl-2-hydroxy-4-methylhex-3-en-5-ynoic acid?



(B)

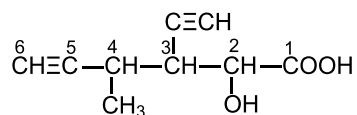


(D)



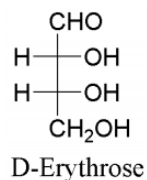
Answer Key (D)

Sol:

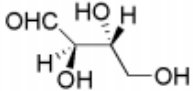
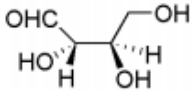
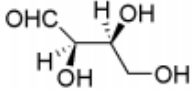
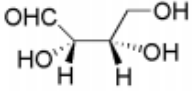


3-ethynyl-2-hydroxy-4-methylhex-3-en-5ynoic acid

6. The Fischer projection of D-erythrose is shown below:



D-Erythrose and its isomers are listed as **P**, **Q**, **R** and **S** in **Column-I**. Choose the correct relationship of **P**, **Q**, **R** and **S** with D-erythrose from **Column-II**.

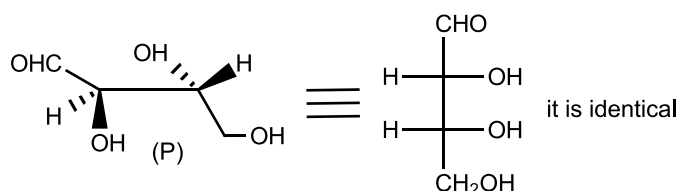
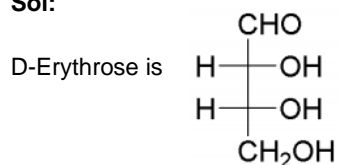
Column-I	Column-II
<p>P. </p> <p>Q. </p> <p>R. </p> <p>S. </p>	<p>1. Diastereomer</p> <p>2. Identical</p> <p>3. Enantiomer</p>

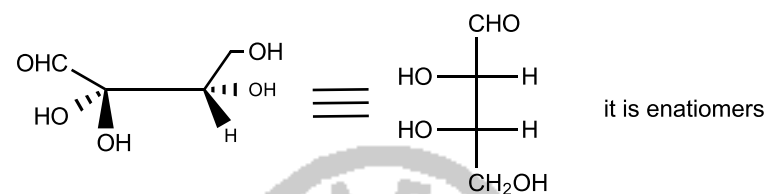
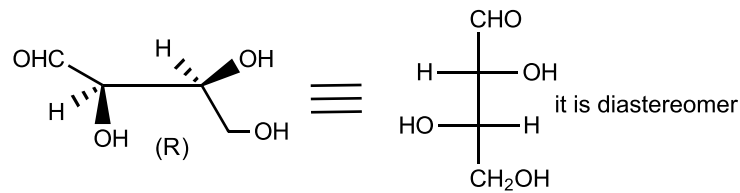
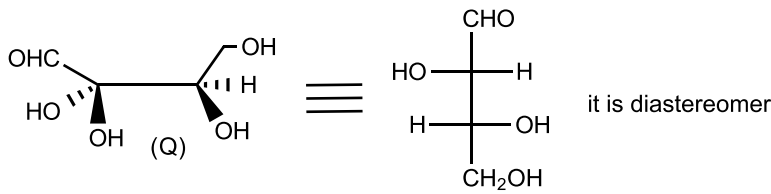
- (A) P → 2, Q → 3, R → 2, S → 2
 (C) P → 2, Q → 1, R → 1, S → 3

- (B) P → 3, Q → 1, R → 1, S → 2
 (D) P → 2, Q → 3, R → 3, S → 1

Answer Key (C)

Sol:





P → 2, Q → 1, R → 1, S → 3

SECTION 2 (Maximum Marks 24)

- This section contains **SIX** (06) questions
- Each question **FOUR** option. **ONLY ONE OR MORE THAN** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme.

<i>Full Marks</i>	:	+4	If only (all) the correct option(s) is (are) chosen;
<i>Partial Marks</i>	:	+3	If all the four options are correct but ONLY three options are chosen;
<i>Partial Marks</i>	:	+2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
<i>Partial Marks</i>	:	+1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
<i>Zero Marks</i>	:	0	If none of the options is chosen (i.e., the questions is unanswered);
<i>Negative Marks</i>	:	-2	In all other cases.

7. In Thermodynamics, the P–V work done is given by

$$w = - \int dV P_{\text{ext}}$$

For a system undergoing a particular process, the work done is,

$$w = - \int dV \left(\frac{RT}{V-b} - \frac{a}{V^2} \right)$$

This equation is applicable to a

- (A) system that satisfies the van der Waals equation of state
 (B) process that is reversible and isothermal
 (C) process that is reversible and adiabatic
 (D) process that is irreversible and at constant pressure

Answer Key (A, B, C)

Sol:

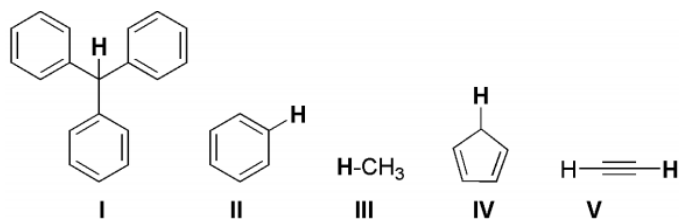
It is given that

$$w = - \int P_{\text{ext}} \cdot dV = - \int PdV = - \int \left(\frac{RT}{V-b} - \frac{a}{V^2} \right) dV$$

When $P_{\text{ext}} \cong P$, the process becomes reversible

$P = \frac{RT}{V-b} - \frac{a}{V^2}$ is the van der Waals equation for real gases and it is applicable for all reversible processes.

8. With respect to the compounds I-V, choose the correct statement(s)

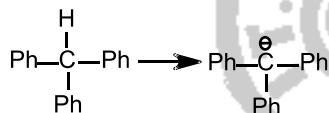


- (A) The acidity of compound I is due to delocalization in the conjugate base
 (B) The conjugate base of compound IV is aromatic
 (C) Compound II becomes more acidic, when it has a $-\text{NO}_2$ substituent
 (D) The acidity of compounds follows the order $\text{I} > \text{IV} > \text{V} > \text{II} > \text{III}$

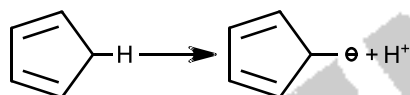
Answer Key (A, B, C)

Sol:

The negative charge in the conjugate base of compound I is stabilized by delocalization due to the presence of three benzene rings in conjugation with carbon carrying negative charge.



The conjugate base of compound IV is cyclopentadienyl anion which is aromatic

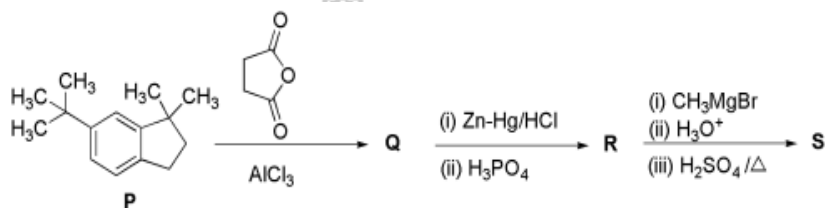


$-\text{NO}_2$ group is electron withdrawing. It increases acid strength

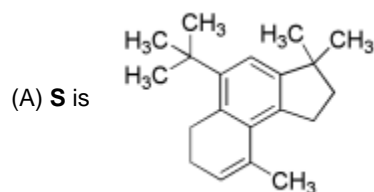
The order of acidic character of the given compounds

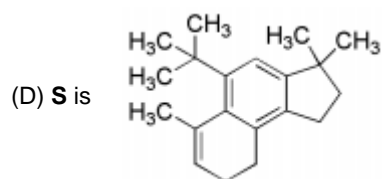
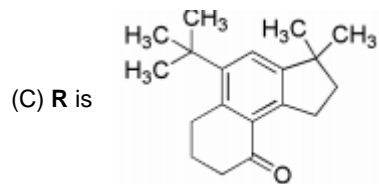
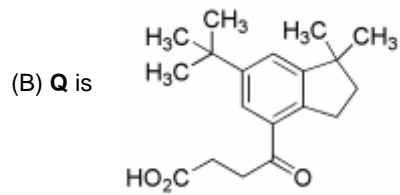
$\text{I} > \text{IV} > \text{V} > \text{II} > \text{III}$

9. In the reaction scheme shown below, Q, R, and S are the major products.



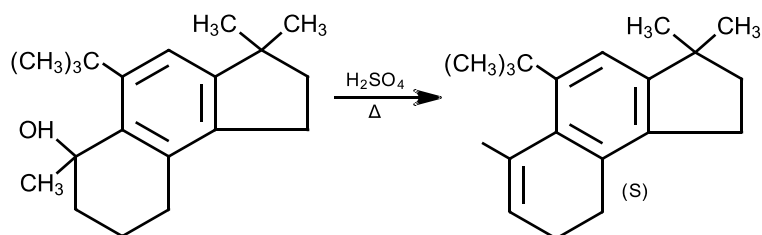
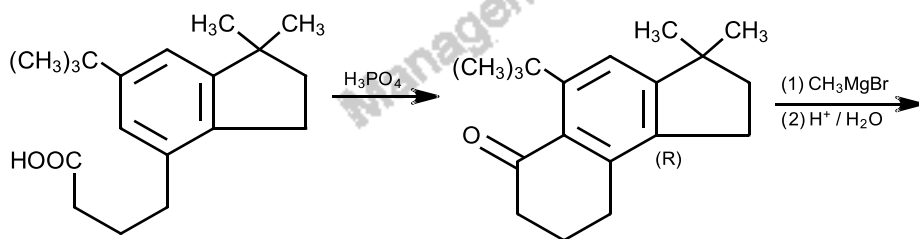
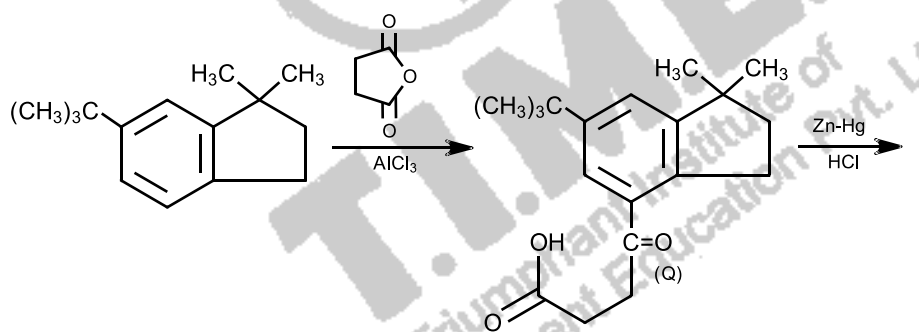
The correct structure of





Answer Key (B, D)

Sol:



10. Choose the correct statement(s) among the following:
- (A) $[\text{FeCl}_4]^-$ has tetrahedral geometry
 (B) $[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$ has 2 geometrical isomers
 (C) $[\text{FeCl}_4]^-$ has higher spin-only magnetic moment than $[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$
 (D) The cobalt ion in $[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$ has sp^3d^2 hybridization

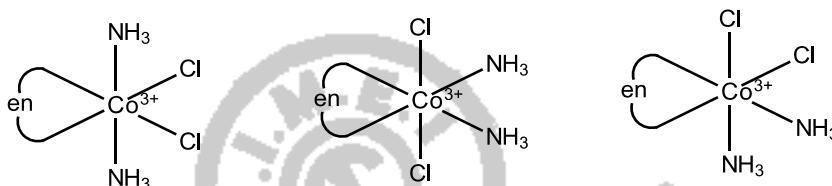
Answer Key (A, C)

Sol:

In $[\text{FeCl}_4]^-$ Fe is in +3 oxidation state (3d^5 configuration)
 Since the ligand field is weak the geometry will be tetrahedral
 Here $\Delta_t < P$ therefore 'd' subshell splits as $e^2 t^3$
 No. of unpaired electron = 5

Magnetic moment $\mu = \sqrt{35} = 5.92 \text{ BM}$

In $[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$ Co is in +3 oxidation state (3d^6 configuration)
 Since the ligand field is strong. Therefore hybridization is d^2sp^3
 Here $\Delta_0 > P \therefore$ subshell splits as $t_{2g}^6 e_g^0$
 Since there is no unpaired electron magnetic moment is zero
 $[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$ has three geometrical isomers

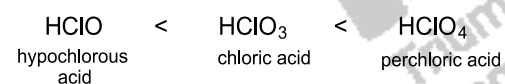


11. Which respective hypochlorite, chlorate and perchlorate ions, choose the correct statement(s)
- (A) The hypochlorite ion is the strongest conjugate base
 (B) The molecular shape of only chlorate ion is influenced by the lone pair of electrons of Cl
 (C) The hypochlorite and chlorate ions disproportionate to give rise to identical set of ions
 (D) The hypochlorite ion oxidizes the sulfite ion

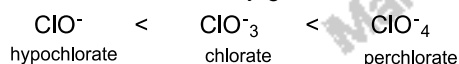
Answer Key (A, B, D)

Sol:

The increase in acidic character of oxyacids



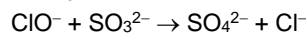
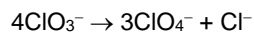
Basic character of conjugate base



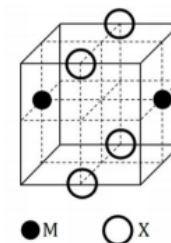
Since hypochlorite ion is a diatomic species it is linear and shape does not get influenced by the presence of lone pair

Chlorine atom in ClO_3^- is sp^3 hybridised with a lone pair occupying one hybridized orbital. So its shape is influenced by the lone pair

ClO_4^- is tetrahedral



12. The cubic unit cell structure of a compound containing cation **M** and anion **X** is shown below. When compared to anion, the cation has smaller ionic radius. Choose the correct statement(s).



- (A) The empirical formula of the compound is MX
 (B) The cation **M** and anion **X** have different coordination geometries
 (C) The ratio of **M-X** bond length to the cubic unit cell edge lengths is 0.866
 (D) The ratio of the ionic radii of cation **M** to anion **X** is 0.414

Answer (A, C)

Sol:

M is at the face centre and X at the edge centre

$$Z_M = \frac{1}{2} \times 2 = 1$$

$$Z_X = \frac{1}{4} \times 4 = 1$$

∴ Formula is MX

Coordination number of **M** and **X** must be the same

$$\text{M-X bond length} = \text{Distance between nearest edge centre to face centre} = \sqrt{3} \frac{a}{2}$$

$$\text{Ratio of M-X bond length to edge length 'a'} = \frac{\sqrt{3}}{2} : 1 = 0.866$$

$$r_M : r_X = 0.732 \text{ (Octahedral void)}$$

SECTION 3 (Maximum Marks 24)

- This section contains **SIX** (06) questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate / round-off** the value of **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme.
Full Marks : +4 If **ONLY** the correct numerical value is entered.
Zero Marks : 0 In all other cases.

13. 5.00 mL of 0.10 M oxalic acid solution taken in a conical flask is titrated against NaOH from a burette using phenolphthalein indicator. The volume of NaOH required for the appearance of permanent faint pink color is tabulated below for the five experiments. What is the concentration, in molarity, of the NaOH solution?

Exp. No.	Vol. of NaOH (mL)
1	12.5
2	10.5
3	9.0
4	9.0
5	9.0

Answer Key (0.11)

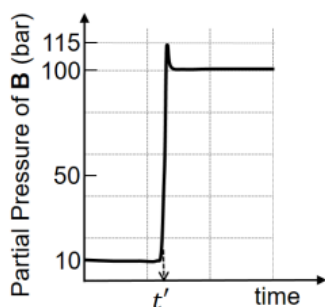
Sol:

$$V_1N_1 = V_2N_2 \text{ (Neutralization)}$$

$$5 \times 0.2 = 9 \times N$$

$$N = \frac{1}{9} = 0.11 = M \text{ (NaOH)}$$

14. Consider the reaction $A \rightleftharpoons B$ at 1000 K. At time t' , the temperature of the system was increased to 2000 K and the system was allowed to reach equilibrium. Throughout this experiment the partial pressure of A was maintained at 1 bar. Given below is the plot of the partial pressure of B with time. What is the ratio of the standard Gibbs energy of the reaction at 1000 K to that at 2000 K



Answer Key (0.25)

Sol:

$$K_p = \frac{p_B}{p_A}$$

$$K_{p_1} = \frac{10}{1} = 10$$

$$K_{p_2} = \frac{100}{1} = 100$$

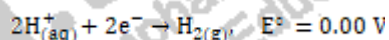
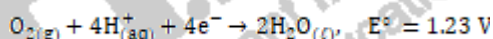
$$\frac{\Delta G_1}{\Delta G_2} = \frac{-RT_1 \ln K_{p_1}}{-RT_2 \ln K_{p_2}} = \frac{1 \log 10}{2 \log 100} = \frac{1}{4} = 0.25$$

15. Consider a 70% efficient hydrogen-oxygen fuel cell working under standard condition at 1 bar and 298 K. Its cell reaction is



The work derived from the cell on the consumption of 1.0×10^{-3} mol of $\text{H}_{2(g)}$ is used to compress 1.00 mol of a monoatomic ideal gas in a thermally insulated container. What is the change in the temperature (in K) of the ideal gas?

The standard reduction potentials for the two half-cells are given below



Use $F = 96500 \text{ C mol}^{-1}$, $R = 8.314 \text{ J mol}^{-1} \text{ K}^{-1}$

Answer Key (13.32)

Sol:

$$E^\circ_{\text{cell}} = 1.23 - 0.00 = 1.23 \text{ V}$$

$$\text{Work done} = -\Delta G^\circ = nFE^\circ$$

$$= 2 \times 10^{-3} \times 96500 \times 1.23 \text{ J}$$

$$\text{Work derived from the cell} = 2 \times 10^{-3} \times 96500 \times 1.23 \times 0.7 \text{ J} \\ \text{(70\% efficiency)}$$

For a thermally insulated container, $q = 0$

$$\therefore w = \Delta U = nC_v \Delta T$$

$$= 1 \times \frac{3}{2} R \times \Delta T$$

Equating

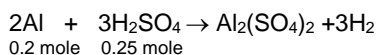
$$2 \times 10^{-3} \times 96500 \times 1.23 \times 0.7 = 1 \times \frac{3}{2} \times 8.314 \times \Delta T$$

$$\Delta T = 13.32$$

16. Aluminium reacts with sulfuric acid to form aluminium sulfate and hydrogen. What is the volume of hydrogen gas in litres (L) produced at 300 K and 1.0 atm pressure, when 5.4 g of aluminium and 50.0 mL of 5.0 M sulfuric acid are combined for the reaction?
(Use molar mass of aluminium as 27.0 g mol⁻¹, R = 0.082 atm L mol⁻¹ K⁻¹)

Answer Key (6.15)

Sol:



0.2 mole 0.25 mole

H₂SO₄ is the limiting reactant

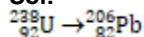
∴ No. of moles of H₂ ↑ produced = 0.25

$$V = \frac{nRT}{P} = \frac{0.25 \times 0.082 \times 300}{1} = 6.15 \text{ L}$$

17. ²³⁸₉₂U is known to undergo radioactive decay to form ²⁰⁶₈₂Pb by emitting alpha and beta particles. A rock initially contained 68 × 10⁻⁶ g of ²³⁸₉₂U. If the number of alpha particles that it would emit during its radioactive decay of ²³⁸₉₂U to ²⁰⁶₈₂Pb in three half-lives is Z × 10¹⁸, then what is the value of Z?

Answer Key (1.20)

Sol:



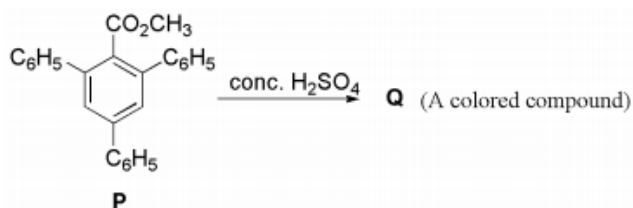
$$\text{No. of atoms of U} = \frac{68 \times 10^{-6}}{238} \times 6 \times 10^{23}$$

For the disintegration of each U to Pb, no. of α particles emitted = $\frac{32}{4} = 8$

After 3 half-lives, no. of atoms disintegrated = $1 - \frac{1}{8} = \frac{7}{8}$ of the initial

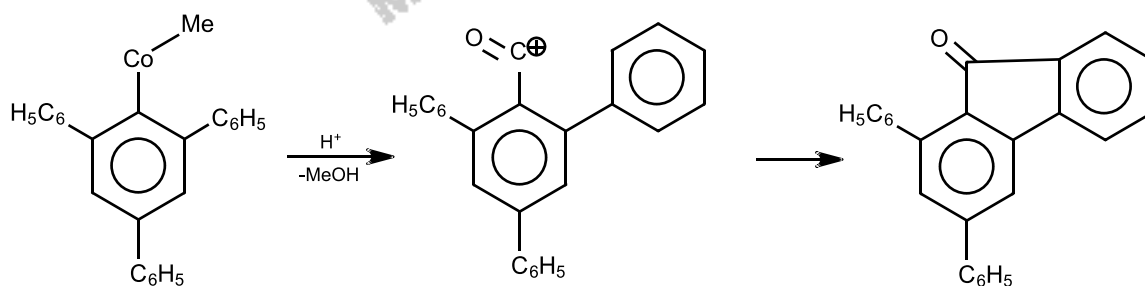
$$\therefore \text{Total no. of } \alpha\text{-particles emitted} = \frac{7}{8} \times 8 \times \frac{68}{238} \times 10^{-6} \times 6 \times 10^{23} = 1.2 \times 10^{18}$$

18. In the following reaction, compound Q is obtained from compound P via an ionic intermediate.



What is the degree of unsaturation of Q?

Answer Key (18.00)



Sol:

Each benzene ring has four double bond equivalents.

Five membered ring gives one double bond equivalent

Carbonyl group gives one double bonds equivalent

∴ The degree of unsaturation = 18

[PART C - MATHEMATICS]

SECTION 1 (Maximum Marks 18)

- This section contains **SIX** (06) questions
 - Each question **FOUR** option. **ONLY ONE** of these four option is the correct answer.
 - For each question, choose the option corresponding to the correct answer.
 - Answer to each question will be evaluated according to the following marking scheme.
- Full Marks* : +3 If **ONLY** the correct options is chosen;
Zero Marks : 0 If none of the options is chosen (i.e., the questions is unanswered);
Negative Marks : -1 In all other cases.

1. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of $ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$ is
- (A) 0 (B) 8000 (C) 8080 (D) 16000

Answer (D)

Sol:

$$\begin{aligned}
 & ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d) \\
 &= a^2(c + d) - a(c^2 + d^2) + b^2(c + d) - b(c^2 + d^2) \\
 &= (a^2 + b^2)(c + d) - (a + b)(c^2 + d^2) \\
 &= \{(a + b)^2 - 2ab\}(c + d) - (a + b)\{(c + d)^2 - 2cd\} \\
 &= 16000
 \end{aligned}$$

2. If the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = |x|(x - \sin x)$, then which of the following statement is **TRUE**?
- (A) f is one-one, but **NOT** onto (B) f is onto, but **NOT** one-one
 (C) f is **BOTH** one-one and onto (D) f is **NEITHER** one-one **NOR** onto

Answer (C)

Sol:

$f(x) = |x|(x - \sin x)$ is odd function because $f(-x) = -f(x)$

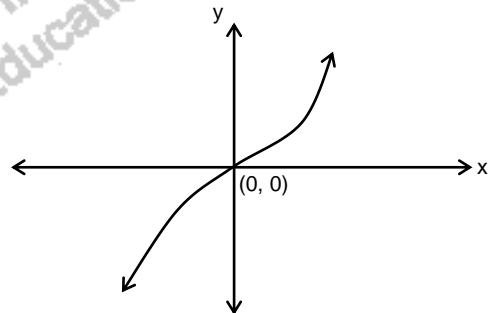
$f(x) = x^2 - x \sin x, x \geq 0$

$f'(x) = 2x - x \cos x - \sin x$

$f'(x) = (x - \sin x) + x(1 - \cos x) > 0$

\therefore graph of $y = f(x)$ is

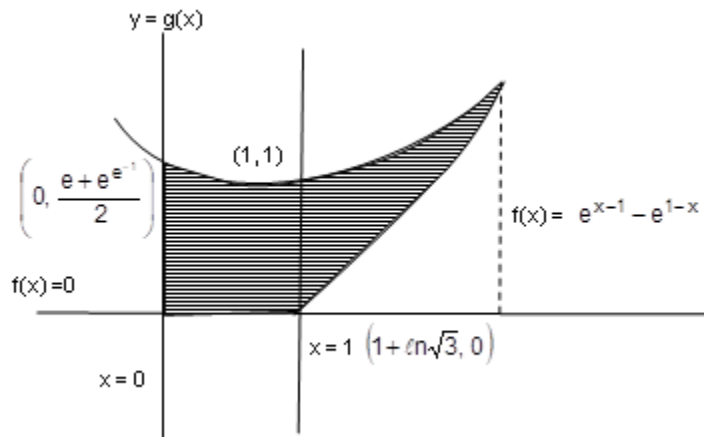
One-one & onto.



3. Let the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = e^{x-1} - e^{|x-1|}$ and $g(x) = \frac{1}{2}(e^{x-1} + e^{1-x})$. Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is
- (A) $(2 - \sqrt{3}) + \frac{1}{2}(e - e^{-1})$ (B) $(2 + \sqrt{3}) + \frac{1}{2}(e - e^{-1})$
 (C) $(2 - \sqrt{3}) + \frac{1}{2}(e + e^{-1})$ (D) $(2 + \sqrt{3}) + \frac{1}{2}(e + e^{-1})$

Answer (A)

Sol:



$$\begin{aligned}
 &= \int_0^1 g(x) dx + \int_1^{1+\ln\sqrt{3}} \{g(x) - f(x)\} dx \\
 &= \int_0^1 \frac{1}{2}(e^{x-1} + e^{1-x}) dx + \int_1^{1+\ln\sqrt{3}} \left\{ \frac{1}{2}(e^{x-1} + e^{1-x}) - (e^{x-1} - e^{1-x}) \right\} dx \\
 &= \frac{1}{2} \int_0^1 (e^{x-1} + e^{1-x}) dx + \frac{1}{2} \int_1^{1+\ln\sqrt{3}} (3e^{1-x} - e^{x-1}) dx \\
 &= \frac{1}{2} [e^{x-1} - e^{1-x}]_0^1 - \frac{1}{2} [3e^{1-x} + e^{x-1}]_1^{1+\ln\sqrt{3}} \\
 &= \frac{1}{2} (e - e^{-1}) - \frac{1}{2} (2\sqrt{3} - 4) = \frac{e^{-1} - e^{-1}}{2} + 2 - \sqrt{3}
 \end{aligned}$$

4. Let a , b and λ be positive real numbers. Suppose P is an end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point P . If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of the ellipse is
- (A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{2}{5}$

Answer (A)

Sol:

$$y^2 = 4\lambda x$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_A = 1 = m_1$$

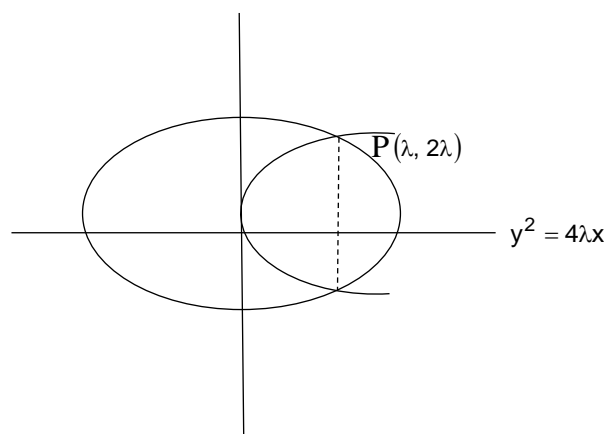
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \left(\frac{dy}{dx}\right)_A = \frac{-b^2}{2a^2} = m_2$$

$$\Rightarrow m_1 m_2 = -1$$

$$\Rightarrow b^2 = 2a^2 \text{ and } a^2 = b^2(1 - e^2)$$

$$1 = 2(1 - e^2)$$

$$e = \frac{1}{\sqrt{2}}$$



5. Let C_1 and C_2 be two biased coins such that the probabilities of getting head in a single toss are $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Suppose α is the number of heads that appear when C_1 is tossed twice. Independently, and suppose β is the number of heads that appear when C_2 is tossed twice, independently. Then the probability that the roots of the quadratic polynomial $x^2 - \alpha x + \beta$ are real and equal, is

- (A) $\frac{40}{81}$ (B) $\frac{20}{81}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

Answer (B)

Sol:

Roots are real & equal

$$\therefore D = 0 \Rightarrow \alpha^2 - 4\beta = 0$$

$$(\alpha = 0, \beta = 0) \text{ or } (\alpha = 2, \beta = 0)$$

$$\begin{aligned} \text{Probability } & {}^2C_0 \left(\frac{1}{2}\right)^2 + {}^2C_0 \left(\frac{2}{3}\right)^2 + {}^2C_2 \left(\frac{2}{3}\right)^2 + {}^2C_1 \left(\frac{1}{3}\right) \left(\frac{2}{3}\right) \\ &= \frac{1}{9} \times \frac{4}{9} + \frac{4}{9} \times \frac{4}{9} = \frac{20}{81} \end{aligned}$$

6. Consider all rectangles lying in the region $\{(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq \frac{\pi}{2} \text{ and } 0 \leq y \leq 2\sin(2x)\}$ and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

- (A) $\frac{3\pi}{2}$ (B) π (C) $\frac{\pi}{2\sqrt{3}}$ (D) $\frac{\pi\sqrt{3}}{2}$

Answer (C)

Sol:

$$2\sin 2\theta_1 = 2\sin 2\theta_2$$

$$2\theta_1 = \pi - 2\theta_2$$

$$\theta_2 = \frac{\pi}{2} - \theta_1$$

$$\text{Perimeter } P(\theta_1, \theta_2) = 2\{(\theta_2 - \theta_1) + 2\sin 2\theta_1\}$$

$$P(\theta_1) = 2\left[\frac{\pi}{2} - 2\theta_1 + 2\sin 2\theta_1\right]$$

$$P'(\theta_1) = 2(-2 + 4\cos 2\theta_1)$$

$$P''(\theta_1) = 2(-8\sin 2\theta_1)$$

for max perimeter

$$P'(\theta_1) = 0 \text{ \& } P''(\theta_1) < 0$$

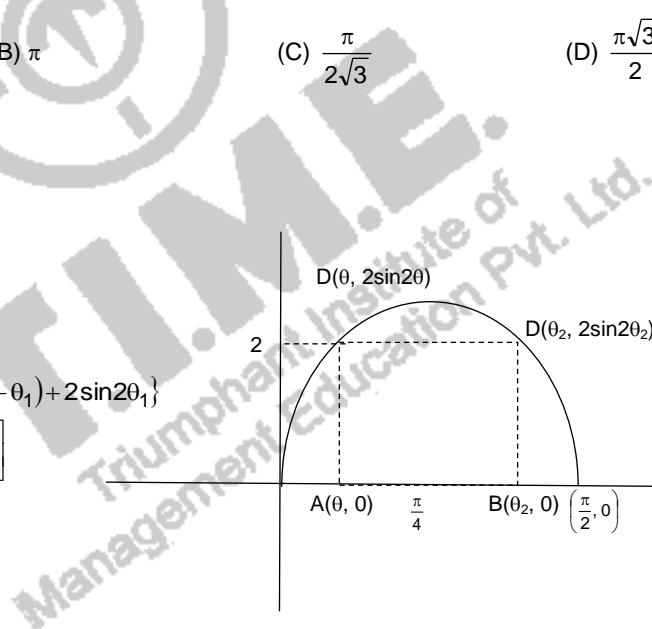
$$\cos 2\theta_1 = \frac{1}{2} \Rightarrow 2\theta_1 = \frac{\pi}{3}, \theta_1 = \frac{\pi}{6}$$

$$\text{Now area at } \theta_1 = \frac{\pi}{6}$$

$$= (\theta_2 - \theta_1) 2\sin 2\theta_1$$

$$= \left(\frac{\pi}{2} - 2\theta_1\right) 2\sin 2\theta_1 = \left(\frac{\pi}{2} - \frac{\pi}{3}\right) 2\sin \frac{\pi}{3}$$

$$= \frac{\pi}{6} \cdot \sqrt{3} = \frac{\pi}{2\sqrt{3}}$$



SECTION 2 (Maximum Marks 24)

- This section contains **SIX** (06) questions
- Each question **FOUR** option. **ONLY ONE OR MORE THAN** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme.

<i>Full Marks</i>	:	+4	If only (all) the correct option(s) is (are) chosen:
<i>Partial Marks</i>	:	+3	If all the four options are correct but ONLY three options are chosen;
<i>Partial Marks</i>	:	+2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
<i>Partial Marks</i>	:	+1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
<i>Zero Marks</i>	:	0	If none of the options is chosen (i.e., the questions is unanswered);
<i>Negative Marks</i>	:	-2	In all other cases.

7. Let the function $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^3 - x^2 + (x-1)\sin x$ and let $g: \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary function. Let $fg: \mathbb{R} \rightarrow \mathbb{R}$ be the product function defined by $(fg)(x) = f(x)g(x)$. Then which of the following statements is/are TRUE?
- (A) If g is continuous at $x = 1$, then fg is differentiable at $x = 1$
 (B) If fg is differentiable at $x = 1$, then g is continuous at $x = 1$
 (C) If g is differentiable at $x = 1$, then fg is differentiable at $x = 1$
 (D) If fg is differentiable at $x = 1$, then g is differentiable at $x = 1$

Answer (A, C)

Sol:

Let $g(x)$ be continuous at $x = 1$

$$\phi = f \circ g = (x-1)(x^2 + \sin x) \cdot g(x)$$

$$\phi'(1^+) = \lim_{h \rightarrow 0} \left[\frac{\phi(1+h) - \phi(1)}{h} \right]$$

$$= \lim_{h \rightarrow 0} \left[\frac{h \cdot \{(1+h)^2 + \sin(1+h)\}g(1+h) - 0}{h} \right]$$

$$= (1 + \sin 1)g(1)$$

$$\phi'(1^-) = \lim_{h \rightarrow 0} \left[\frac{\phi(1-h) - \phi(1)}{-h} \right]$$

$$= \lim_{h \rightarrow 0} \left[\frac{(-h) \cdot \{(1-h)^2 + \sin(1-h)\}g(1-h)}{-h} \right]$$

$$= (1 + \sin 1)g(1)$$

\therefore A is true

If g is differentiable then g is continuous

\therefore C is true

$$\text{Let } g = \frac{1}{(x-1)}$$

In this case $g(x)$ is neither continuous nor differentiable at $x = 1$.

$$\phi = f \cdot g = x^2 + \sin x$$

$f \cdot g$ differentiable at $x = 1$

but g is neither continuous nor differentiable: B and D are false

8. Let M be a 3×3 invertible matrix with real entries and let I denote the 3×3 identity matrix. If $M^{-1} = \text{adj}(\text{adj}M)$, then which of the following statements is/are ALWAYS TRUE?
 (A) $M = I$ (B) $\det M = 1$ (C) $M^2 = I$ (D) $(\text{adj } M)^2 = I$

Answer (B, C, D)

Sol:

$$M^{-1} = \text{adj}(\text{adj}M)$$

$$M^{-1} = |M|^{3-2} \cdot M \left[\because \text{adj}(\text{adj}M) = |M|^{n-2} \cdot M \right]$$

$$\Rightarrow M^{-1} = |M| \cdot M \rightarrow (1)$$

$$\Rightarrow M \cdot M^{-1} = |M| \cdot M^2 \Rightarrow I = |M| \cdot M^2$$

$$\Rightarrow M^2 = \frac{1}{|M|} \cdot I \rightarrow (2)$$

$$\text{From (1) } |M^{-1}| = ||M| \cdot M|$$

$$\Rightarrow \frac{1}{|M|} = |M|^3 \cdot |M| \Rightarrow |M|^5 = 1$$

$$\Rightarrow |M| = 1$$

$$\therefore (2) \Rightarrow M^2 = I \Rightarrow (C) \text{ is true}$$

(B) is true

$$\text{adj}(M^2) = \text{adj}(I)$$

$$\Rightarrow (\text{adj}M)^2 = I \left[\because \text{adj}(M^k) = (\text{adj}M)^k \text{ and } \text{adj}I = I \right]$$

\therefore (D) is true

(A) is false

9. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE?

(A) $\left| z + \frac{1}{2} \right| \leq \frac{1}{2}$ for all $z \in S$

(B) $|z| \leq 2$ for all $z \in S$

(C) $\left| z + \frac{1}{2} \right| \geq \frac{1}{2}$ for all $z \in S$

(D) The set S has exactly four elements

Answer (B, C)

Sol:

$$\left| \left(z + \frac{1}{2} \right)^2 + \frac{3}{4} \right| = 1 \leq \left| z + \frac{1}{2} \right|^2 + \frac{3}{4}$$

$$\frac{1}{4} \leq \left| z + \frac{1}{2} \right|^2 \Rightarrow \left| z + \frac{1}{2} \right| \geq \frac{1}{2}$$

$$\Rightarrow (C) \text{ is true} \quad (A) \text{ is false}$$

$$\text{also, } \left| \left(z + \frac{1}{2} \right)^2 + \frac{3}{4} \right| \geq \left| \left| z + \frac{1}{2} \right|^2 - \frac{3}{4} \right|$$

$$\Rightarrow \left| \left| z + \frac{1}{2} \right|^2 - \frac{3}{4} \right| \leq 1$$

$$\Rightarrow \left| z + \frac{1}{2} \right| \leq \frac{\sqrt{9}}{2}$$

\therefore (B) is true

$$\text{since, } z^2 + z + 1 = \cos\theta + i\sin\theta$$

$$\Rightarrow z^2 + z + [(1 - (\cos\theta + i\sin\theta))] = 0$$

Where $\theta \in \mathbb{R}$

$\Rightarrow z$ has infinite number of solution

\therefore D is false

10. Let x, y and z be positive real numbers. Suppose x, y and z are the lengths of the sides of a triangle opposite to its angles X, Y and Z , respectively. If $\tan \frac{X}{2} + \tan \frac{Z}{2} = \frac{2y}{x+y+z}$, then which of the following statements is/are TRUE?
- (A) $2Y = X + Z$ (B) $Y = X + Z$ (C) $\tan \frac{X}{2} = \frac{x}{y+z}$ (D) $x^2 + z^2 - y^2 = xz$

Answer (B, C)

Sol:

For our convenience $X = A, Y = B$ and $Z = C \Rightarrow x = a, y = b$ and $z = c$

Given that,

$$\tan \frac{A}{2} + \tan \frac{C}{2} = \frac{2b}{a+b+c}$$

$$\Rightarrow \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{2b}{2s}$$

$$\Rightarrow \sqrt{\frac{s-b}{s}} \left[\frac{\sqrt{s-c}}{\sqrt{s-a}} + \sqrt{\frac{s-a}{s-c}} \right] = \frac{b}{s}$$

$$\Rightarrow \frac{\sqrt{s-b}}{\sqrt{s}} \left[\frac{(s-c) + (s-a)}{\sqrt{s-a} \cdot \sqrt{s-c}} \right] = \frac{b}{s}$$

$$\Rightarrow \frac{\sqrt{s-b}}{\sqrt{s}} \cdot \frac{b}{\sqrt{s-a} \cdot \sqrt{s-c}} = \frac{b}{s}$$

$$\sqrt{\frac{(s-b)}{(s-a)(s-c)}} = \frac{1}{\sqrt{s}}$$

$$\Rightarrow \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = 1 \Rightarrow \cot \frac{B}{2} = 1$$

$$\tan \frac{B}{2} = 1 \Rightarrow B = 90^\circ$$

$$\Rightarrow A + C = 90^\circ$$

$$\Rightarrow A + C = B \Rightarrow Y = X + Z$$

(B) is true

(A) is false

(D) is false

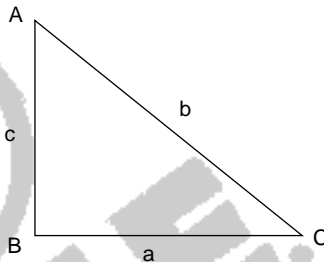
Since $x^2 + z^2 - y^2$

$$= a^2 + c^2 - b^2 = b^2 - b^2 = 0 \neq xz$$

$$\frac{x}{y+z} = \frac{a}{b+c} = \frac{\sin A}{\sin B + \sin C} = \frac{\sin A}{1 + \cos A}$$

$$= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \cos^2 \frac{A}{2}} = \tan \frac{A}{2} = \tan \frac{X}{2}$$

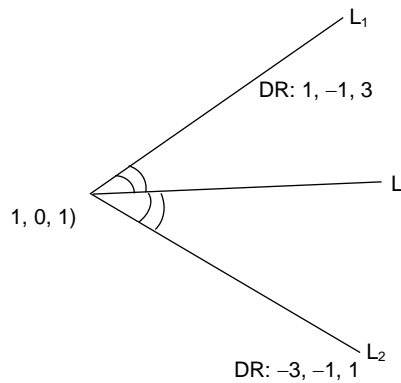
C is true



11. Let L_1 and L_2 be the following straight lines. $L_1 : \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3}$ and $L_2 : \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}$. Suppose the straight line $L : \frac{x-\alpha}{\ell} = \frac{y-1}{m} = \frac{z-\gamma}{-2}$ lies in the plane containing L_1 and L_2 and passes through the point of intersection of L_1 and L_2 . If the line L bisects the acute angle between the lines L_1 and L_2 , then which of the following statements is/are TRUE?
- (A) $\alpha - \gamma = 3$ (B) $\ell + m = 2$ (C) $\alpha - \gamma = 1$ (D) $\ell + m = 0$

Answer (A, B)

Sol:



'L' passes through \$(1, 0, 1)\$

$$\Rightarrow \frac{1-\alpha}{\ell} = \frac{0-1}{m} = \frac{1-\gamma}{-2} \rightarrow (1)$$

Let \$\vec{a} = \hat{i} - \hat{j} + 3\hat{k}\$ (along \$L_1\$)

\$\vec{b} = -3\hat{i} - \hat{j} + \hat{k}\$ (along \$L_2\$)

The vector along L is \$\lambda \left[\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|} \right]\$

$$= \lambda \left[\frac{(\hat{i} - \hat{j} + 3\hat{k})}{\sqrt{10}} + \frac{(-3\hat{i} - \hat{j} + \hat{k})}{\sqrt{10}} \right]$$

$$= \frac{\lambda}{\sqrt{10}} [-2\hat{i} - 2\hat{j} + 4\hat{k}] = \frac{-2\lambda}{\sqrt{10}} (\hat{i} + \hat{j} - 2\hat{k})$$

$$= k(\hat{i} + \hat{j} - 2\hat{k}) = \ell\hat{i} + m\hat{j} - 2\hat{k}$$

$$\Rightarrow k = 1 \Rightarrow \ell = 1, m = 1$$

$$\therefore \ell + m = 2$$

(B) is true

(D) is false

$$(1) \Rightarrow \frac{1-\alpha}{1} = \frac{-1}{1} = \frac{1-\gamma}{-2}$$

$$\Rightarrow 1-\alpha = -1 \Rightarrow \alpha = 2$$

$$\text{and } \frac{1-\gamma}{-2} = -1 \Rightarrow 1-\gamma = 2 \Rightarrow \gamma = -1$$

$$\Rightarrow \alpha - \gamma = 3$$

\$\therefore\$ (A) is true

(C) is false

12. Which of the following inequalities is/are TRUE?

(A) \$\int_0^1 x \cos x dx \geq \frac{3}{8}\$

(B) \$\int_0^1 x \sin x dx \geq \frac{3}{10}\$

(C) \$\int_0^1 x^2 \cos x dx \geq \frac{1}{2}\$

(D) \$\int_0^1 x^2 \sin x dx \geq \frac{2}{9}\$

Answer (A, B, D)

Sol:

If \$x > 0\$

$$\cos x > 1 - \frac{x^2}{2!} \text{ and } \sin x > x - \frac{x^3}{3!}$$

(approximately equal for small positive values of \$x\$)

$$\int_0^1 x \cos x dx > \int_0^1 x \left(1 - \frac{x^2}{2}\right) dx$$

$$\text{RHS} = \left[\frac{x^2}{2} - \frac{1}{2} \cdot \frac{x^4}{4} \right]_0^1 = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

$$\therefore \int_0^1 x \cos x > \frac{3}{8} \Rightarrow \text{(A) is true}$$

$$\int_0^1 x \sin x > \int_0^1 x \left(x - \frac{x^3}{3!}\right) dx$$

$$\text{RHS} = \left[\frac{x^3}{3} - \frac{1}{6} \cdot \frac{x^5}{5} \right]_0^1 = \frac{1}{3} - \frac{1}{30} = \frac{9}{30} = \frac{3}{10}$$

$$\therefore \int_0^1 x \sin x > \frac{3}{10} \quad \text{(B) is true}$$

$$\cos x < 1$$

$$\int_0^1 x^2 \cos x < \int_0^1 x^2 \cdot 1 dx$$

$$\text{RHS} = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$

$$\therefore \int_0^1 x^2 \cos x < \frac{1}{3} \quad \left(\text{Hence Not greater than } \frac{1}{2} \right)$$

\therefore (C) is false

$$\int_0^1 x^2 \sin x dx > \int_0^1 x^2 \left(x - \frac{x^2}{3!}\right) dx$$

$$\text{RHS} = \left[\frac{x^4}{4} - \frac{1}{6} \cdot \frac{x^6}{6} \right]_0^1 = \frac{1}{4} - \frac{1}{36} = \frac{8}{36} = \frac{2}{9}$$

$$\therefore \int_0^1 x^2 \sin x > \frac{2}{9} \Rightarrow \text{(D) is true}$$

SECTION 3 (Maximum Marks 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate / round-off** the value of **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks	:	+4	If ONLY the correct numerical value is entered.
Zero Marks	:	0	In all other cases.

13. Let m be the minimum possible value of $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$ where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is _____.

Answer (8)

Sol:

Let $3^{y_1}, 3^{y_2}, 3^{y_3}$ be three numbers

We know that $AM \geq GM$

$$\Rightarrow \frac{3^{y_1} + 3^{y_2} + 3^{y_3}}{3} \geq (3^{y_1} \cdot 3^{y_2} \cdot 3^{y_3})^{1/3}$$

$$\Rightarrow 3^{y_1} + 3^{y_2} + 3^{y_3} \geq 3 \cdot (3^{y_1+y_2+y_3})^{1/3} = 81$$

Applying log on both sides

$$\log_3(3^{y_1} + 3^{y_2} + 3^{y_3}) \geq \log_3 81$$

$$\Rightarrow 3^{y_1} + 3^{y_2} + 3^{y_3} \geq 4 = m \rightarrow (1)$$

Let x_1, x_2, x_3 , be three numbers

$$A.M \geq G.M \Rightarrow \frac{x_1 + x_2 + x_3}{3} \geq (x_1 x_2 x_3)^{1/3}$$

$$\Rightarrow x_1 x_2 x_3 \leq 27$$

Applying log m both sides

$$\log_3(x_1 x_2 x_3) \leq \log_3 27$$

$$\log_3 x_1 + \log_3 x_2 + \log_3 x_3 \leq 3 = M$$

Therefore, $\log_2 m^3 + \log_3 M^2$

$$\log_2 64 + \log_3 9$$

$$= 6 + 2 = 8$$

14. Let a_1, a_2, a_3, \dots be a sequence of positive integers in arithmetic progression with common difference 2. Also, let b_1, b_2, b_3, \dots be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c , for which the equality $2(a_1 + a_2 + \dots + a_n) = b_1 + b_2 + \dots + b_n$ holds for some positive integer n , is _____

Answer (1)

Sol:

Given $a_1 = b_1 = c$

Common difference of AP = 2

Common ratio of G.P = 2

$$2(a_1 + a_2 + \dots + a_n) = b_1 + b_2 + \dots + b_n$$

$$\Rightarrow 2(c + c + 2 + \dots + c + (n-1)2) = c + 2c + 4c + 2^{n-1}c$$

$$\Rightarrow 2\left(nc + \frac{2(n-1)n}{2}\right) = c(2^n - 1)$$

$$2n^2 - 2n = c(2^n - 2n - 1) \Rightarrow c = \frac{2n^2 - 2n}{2^n - 2n - 1}$$

Now c is a positive integer.

When $n = 1$	$c = 0$	$n = 4$	$c = 3.42$
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$n = 2$	$c = -4$
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$n = 3$	$c = 12$
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since c is a positive integer for $n = 3$ only, the required answer is 1

15. Let $f : [0, 2] \rightarrow \mathbb{R}$ be the function defined by $f(x) = (3 - \sin(2\pi x))\sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right)$. If $\alpha, \beta \in [0, 2]$ sre such that $\{x \in [0, 2]: f(x) \geq 0\} = [\alpha, \beta]$ is _____

Answer (1)

Sol:

putting $\pi x - \frac{\pi}{4} = t$

$$\Rightarrow x = \frac{1}{\pi} \left(t + \frac{\pi}{4} \right)$$

$$f(x) = (3 - \sin 2\pi x) \sin \left(\pi x - \frac{\pi}{4} \right) - \sin \left(3\pi x + \frac{\pi}{4} \right)$$

$$\Rightarrow f(x) = \left(3 - \sin \left(2t + \frac{\pi}{2} \right) \right) \sin t - \sin \left(3t + \frac{3\pi}{4} + \frac{\pi}{4} \right)$$

$$\Rightarrow f(x) = 3 \sin t - \cos 2t \sin t - \sin(3t + \pi)$$

$$\Rightarrow f(x) = 3 \sin t - \cos 2t + \sin 3t$$

Given $f(x) \geq 0$

$$\Rightarrow \sin t (3 - (1 - 2\sin^2 t) - (4\sin^2 t - 3)) \geq 0$$

$$\Rightarrow \sin t (5 - 2\sin^2 t) \geq 0 \Rightarrow \sin t \geq 0$$

since $x \in [0, 2]$ $t \in \left[\frac{-\pi}{4}, \frac{7\pi}{4} \right]$

since $\sin t \geq 0$ $t \in [0, \pi]$

$$\Rightarrow x = \left[\frac{1}{4}, \frac{5}{4} \right]$$

$$\Rightarrow \alpha = \frac{1}{4} \quad \beta = \frac{5}{4}$$

$$\Rightarrow \beta - \alpha = \frac{5}{4} - \frac{1}{4} = 1$$

16. In a triangle PQR, let $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $\frac{\vec{a} \cdot (\vec{c} - \vec{b})}{\vec{c} \cdot (\vec{a} - \vec{b})} = \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}$, then the value of $|\vec{a} \times \vec{b}|^2$ is _____

Answer (108)

Sol:

Since PQ is a triangle

$$\vec{a} + \vec{b} + \vec{c} = 0$$

$$|\vec{a}| = 3 \quad |\vec{b}| = 4$$

$$\frac{\vec{a} \cdot (\vec{c} - \vec{b})}{\vec{c} \cdot (\vec{a} - \vec{b})} = \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}$$

$$\Rightarrow \frac{(\vec{b} - \vec{c}) \cdot (\vec{c} - \vec{b})}{(-\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})} = \frac{3}{7}$$

$$\Rightarrow \frac{|\vec{b}|^2 - |\vec{c}|^2}{|\vec{b}|^2 - |\vec{a}|^2} = \frac{3}{7}$$

$$\Rightarrow \frac{16 - |\vec{c}|^2}{16 - 9} = \frac{3}{7}$$

$$\Rightarrow |\vec{c}|^2 = 13$$

$$\Rightarrow |\vec{a} + \vec{b}|^2 = 13$$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b} = 13$$

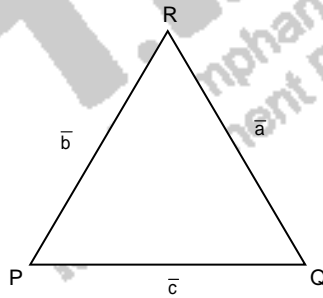
$$9 + 16 + 2\vec{a} \cdot \vec{b} = 13$$

$$\vec{a} \cdot \vec{b} = -6$$

Now, $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$

$$|\vec{a} \times \vec{b}|^2 + 36 = 3^2 \times 4^2$$

$$\Rightarrow |\vec{a} \times \vec{b}|^2 = 144 - 36 = 108$$



17. For a polynomial $g(x)$ with real coefficients, let m_g denote the number of distinct real roots of $f(x)$. Suppose S is the set of polynomials with real coefficients defined by

$$S = \left\{ (x^2 - 1)^2 (a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

For a polynomial f , let f' and f'' denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is _____

Answer (5)

Sol:

$$f(x) = (x^2 - 1)^2 g(x) \text{ where } g(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

$$f(1) = 0; f(-1) = 0$$

\therefore By Rolle's theorem $f'(c) = 0$ for $-1 < c < 1$

Also by actual difference we can see that

$$f'(1) = 0 \text{ \& } f'(-1) = 0$$

$\therefore f'(x) = 0$ has atleast 3 roots

$\Rightarrow f''(x) = 0$ has atleast two roots

Minimum of $(m_{f'} + m_{f''})$ is $3 + 2 = 5$

$$\text{For, } f' = (x^2 - 1)^2 g'(x) + g(x) 2(x^2 - 1)$$

$$= (x^2 - 1) [(x^2 - 1)g'(x) + 2g(x)]$$

$$\Rightarrow f'(1) = 0; f'(-1) = 0$$

18. Let e denote the base of the natural logarithm. The value of the real number a for which the right hand limit

$$\lim_{x \rightarrow 0^+} \frac{(1-x)^{1/x} - e^{-1}}{x^a}$$
 is equal to a nonzero real number, is _____

Answer (1)

Sol:

Putting $x = 0$, we get $\frac{0}{0}$

So applying L Hospital's rule,

$$\frac{(1-x)^{1/x} \left[\frac{x}{1-x} - \ln(1-x) \right]}{x^2}$$

$$\lim_{x \rightarrow 0^+} \frac{x^2}{ax^{2-1}}$$

$$= \lim_{x \rightarrow 0} \frac{(1-x)^{1/x} \left[\frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \right]}{ax^{a-1}(x-1)}$$

This will be non-zero if $a - 1 = 0$

$$\Rightarrow a = 1$$