

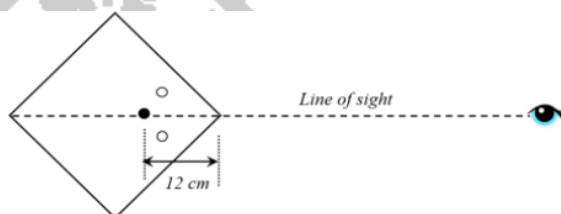
QUESTIONS & KEYS FOR JEE (ADVANCED)-2020 (PAPER 2)
[PHYSICS, CHEMISTRY & MATHEMATICS]

[PART A - PHYSICS]

SECTION 1 (Maximum Marks 18)

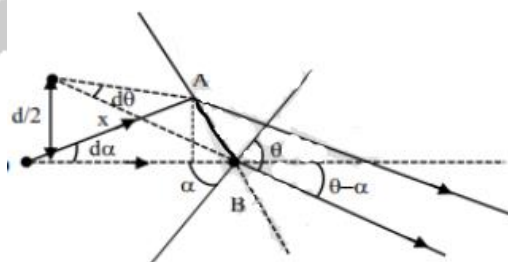
- This section contains **SIX** (06) questions
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, **BOTH INCLUSIVE**.
- For each question, either the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme.
Full Marks : +3 If ONLY the correct integer is entered.
Zero Marks : 0 If the questions is unanswered;
Negative Marks : -1 In all other cases.

1. A large square container with thin transparent vertical walls and filled with water (refractive index $\frac{4}{3}$) is kept on a horizontal table. A student holds a thin straight wire vertically inside the water 12 cm from one of its corners, as shown schematically in the figure. Looking at the wire from this corner, another student sees two images of the wire, located symmetrically on each side of the line of sight as shown. The separation (in cm) between these images is _____.



Answer Key (4)

Sol:



We will assume that observer sees the image of object through edge $\Rightarrow \alpha = 45^\circ$

$$AB = \frac{12d\alpha}{\cos\alpha} = \frac{xd\theta}{\cos\theta}$$

By applying Snell's Law

$$\sin\alpha = 1\sin\theta$$

$$\cos\alpha d\alpha = \cos\theta d\theta$$

$$\Rightarrow \frac{9}{\cos^2\alpha} = \frac{x}{\cos^2\theta}$$

$$1 \sin \theta = \frac{4}{3} \sin \alpha$$

$$\Rightarrow \theta = \frac{2\sqrt{2}}{3} \Rightarrow x = 18 \times \frac{1}{9} = 2$$

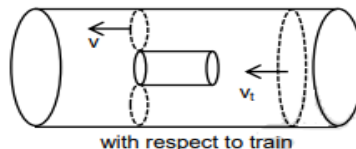
$$d = 2x \sin(\theta - \alpha)$$

$$= 4 \times \frac{1}{\sqrt{3}} \left(\frac{2\sqrt{2}}{3} - \frac{1}{3} \right) = \frac{8-2\sqrt{2}}{3} \approx 1.73 \approx 2$$

2. A train with cross-sectional area S_t is moving with speed v_t inside a long tunnel of cross-sectional area S_0 ($S_0 = 4S_t$). Assume that almost all the air (density ρ) in front of the train flows back between its sides and the walls of the tunnel. Also, the air flow with respect to the train is steady and laminar. Take the ambient pressure and that inside the train to be p_0 . If the pressure in the region between the sides of the train and the tunnel walls is p , then $p_0 - p = \frac{7}{2N} \rho v_t^2$. The value of N is _____.

Answer Key (9)

Sol:



Applying Bernoulli's equation

$$P_0 + \frac{1}{2} \rho v_t^2 = P + \frac{1}{2} \rho v^2$$

$$P_0 - P = \frac{1}{2} \rho (v^2 - v_t^2) \dots \dots (i)$$

From equation of continuity

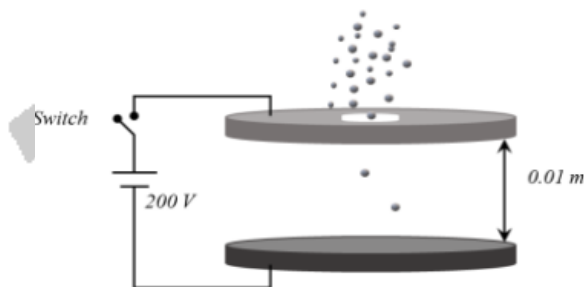
$$\text{Also, } 4S_t v_t = v \times 3S_t \Rightarrow v = \frac{4}{3} v_t \dots \dots (ii)$$

From (i) and (ii)

$$P_0 - P = \frac{1}{2} \rho \left(\frac{16}{9} v_t^2 - v_t^2 \right) = \frac{1}{2} \rho \frac{7v_t^2}{9}$$

$$\therefore N = 9$$

3. Two large circular discs separated by a distance of 0.01 m are connected to a battery via a switch as shown in the figure. Charged oil drops of density 900 kg m^{-3} are released through a tiny hole at the center of the top disc. Once some oil drops achieve terminal velocity, the switch is closed to apply a voltage of 200 V across the discs. As a result, an oil drop of radius $8 \times 10^{-7} \text{ m}$ stops moving vertically and floats between the discs. The number of electrons present in this oil drop is _____. (neglect the buoyancy force, take acceleration due to gravity = 10 ms^{-2} and charge on an electron (e) = $1.6 \times 10^{-19} \text{ C}$)



Answer Key (6)

Sol:

$$E = \frac{V}{d} = \frac{200}{0.01} = 2 \times 10^4 \text{ V/m}$$

When terminal velocity is achieved

$$qE = mg$$

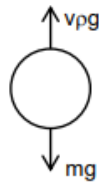
$$\Rightarrow n \times 1.6 \times 10^{-19} \times 2 \times 10^4 = \frac{4\pi}{3} (8 \times 10^{-7})^3 \times 900 \times 10$$

$$\Rightarrow n \approx 6$$

4. A hot air balloon is carrying some passengers, and a few sandbags of mass 1 kg each so that its total mass is 480 kg. Its effective volume giving the balloon its buoyancy is V . The balloon is floating at an equilibrium height of 100 m. When N number of sandbags are thrown out, the balloon rises to a new equilibrium height close to 150 m with its volume V remaining unchanged. If the variation of the density of air with height h from the ground is $\rho(h) = \rho_0 e^{\frac{-h}{h_0}}$, where $\rho_0 = 1.25 \text{ kg m}^{-3}$ and $h_0 = 6000 \text{ m}$, the value of N is _____.

Answer Key (4)

Sol:



$$480 \times g = v\rho_1 g$$

$$(480 - N) g = v\rho_2 g$$

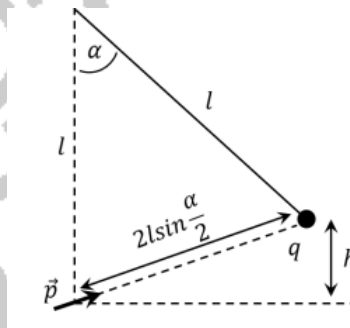
$$\frac{480 - N}{480} = \frac{\rho_2}{\rho_1}$$

$$\left(1 - \frac{N}{480}\right) = \frac{e^{-h_2/-h_0}}{e^{-h_1/h_0}} = e^{\frac{h_1 - h_2}{h_0}} = e^{-\frac{50}{6000}}$$

$$1 - \frac{N}{480} = 1 - \frac{50}{6000} \Rightarrow N = \frac{50 \times 480}{6000} = 4$$

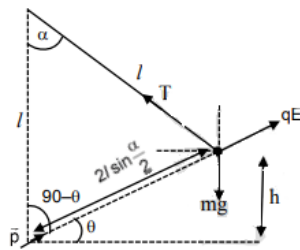
5. A point charge q of mass m is suspended vertically by a string of length l . A point dipole of dipole moment p is now brought towards q from infinity so that the charge moves away. The final equilibrium position of the system including the direction of the dipole, the angles and distances is shown in the figure below. If the work done in bringing the dipole to this position is $N \times (mgh)$, where g is the acceleration due to gravity, then the value of N is _____.

(Note that for three coplanar forces keeping a point mass in equilibrium, $\frac{F}{\sin\theta}$ is the same for all forces, where F is any one of the forces and θ is the angle between the other two forces)



Answer Key (2)

Sol:



$$U_i = 0$$

$$U_f = \frac{kqP}{(2l \sin \frac{\alpha}{2})^2} + mgh \dots\dots(i)$$

Now, from ΔOAB

$$\alpha + 90 - \theta + 90 - \theta = 180$$

$$\Rightarrow \alpha = 2\theta$$

$$\text{From } \Delta ABC : h = 2l \sin \frac{\alpha}{2} \sin \theta$$

$$h = 2l \sin \left(\frac{\alpha}{2}\right) \sin \left(\frac{\alpha}{2}\right)$$

$$\Rightarrow h = 2l \sin^2 \left(\frac{\alpha}{2}\right)$$

Now charge is in equilibrium at point B.

So, using sine rule

$$\Rightarrow \frac{mg}{\sin\left[90 + \frac{\alpha}{2}\right]} = \frac{qE}{\sin[180 + 2\theta]}$$

$$\Rightarrow \frac{mg}{\cos\frac{\alpha}{2}} = \frac{qE}{\sin 2\theta}$$

$$\Rightarrow \frac{mg}{\cos\frac{\alpha}{2}} = \frac{qE}{\sin\alpha} \Rightarrow \frac{mg}{\cos\frac{\alpha}{2}} = \frac{qE}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$\Rightarrow qE = mg 2\sin\left(\frac{\alpha}{2}\right)$$

$$\Rightarrow \frac{q 2kp}{[2\sin\frac{\alpha}{2}]^3} = mg 2\sin\left(\frac{\alpha}{2}\right) \Rightarrow \frac{kpq}{[2\sin\frac{\alpha}{2}]^2}$$

$$= mg \sin\left(\frac{\alpha}{2}\right) \times (2\sin\frac{\alpha}{2})$$

$$\Rightarrow \frac{kpq}{[2\sin\frac{\alpha}{2}]^2} = mgh$$

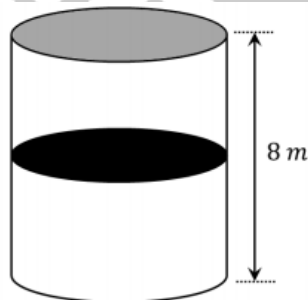
\Rightarrow substituting this in equation (i)

$$U_f = mgh + \frac{kpq}{[2\sin\frac{\alpha}{2}]^2}$$

$$\Rightarrow U_f = 2mgh$$

$$W = \Delta U = Nmgh = N = 2$$

6. A thermally isolated cylindrical closed vessel of height 8 m is kept vertically. It is divided into two equal parts by a diathermic (perfect thermal conductor) frictionless partition of mass 8.3 kg. Thus the partition is held initially at a distance of 4 m from the top, as shown in the schematic figure below. Each of the two parts of the vessel contains 0.1 mole of an ideal gas at temperature 300 K. The partition is now released and moves without any gas leaking from one part of the vessel to the other. When equilibrium is reached, the distance of the partition from the top (in m) will be _____ (take the acceleration due to gravity = 10 ms^{-2} and the universal gas constant = $8.3 \text{ J mol}^{-1}\text{K}^{-1}$).

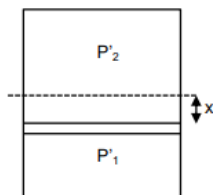


Answer Key (6)

Sol:

Assuming temperature remains constant at 300 K From $P_1V_1 = P_2V_2$

$$\frac{P_1\left(\frac{V_0}{2}\right)}{T} = \frac{P_2\left(\frac{V_0 - Ax}{2}\right)}{T}$$



$$(P_1 - P_2)A = mg$$

$$\left[\frac{P_1\left(\frac{V_0}{2}\right)}{\frac{V_0 - Ax}{2}} - \frac{P_2\left(\frac{V_0}{2}\right)}{\frac{V_0 + Ax}{2}}\right]A = mg$$

$$nRT \left[\frac{1}{4-x} - \frac{1}{4+x}\right] = mg$$

$$3 \left(\frac{2x}{16-x^2}\right) = 1$$

$$6x = 16 - x^2$$

$$x^2 + 6x - 16 = 0$$

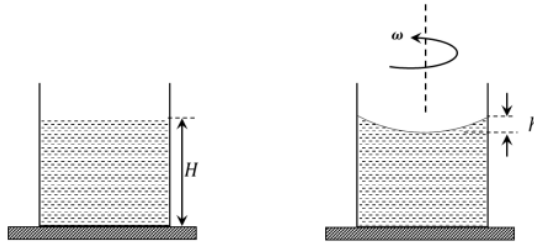
$$x = 2 \text{ distance} = 4 + 2 = 6\text{m}$$

SECTION 2 (Maximum Marks 24)

- This section contains **SIX** (06) questions
- Each question **FOUR** options. **ONE OR MORE THAN** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme.

<i>Full Marks</i>	:	+4	If only (all) the correct option(s) is (are) chosen:
<i>Partial Marks</i>	:	+3	If all the four options are correct but ONLY three options are chosen;
<i>Partial Marks</i>	:	+2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
<i>Partial Marks</i>	:	+1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
<i>Zero Marks</i>	:	0	If none of the options is chosen (i.e., the question is unanswered);
<i>Negative Marks</i>	:	-2	In all other cases.

7. A beaker of radius r is filled with water (refractive index $\frac{4}{3}$) up to a height H as shown in the figure on the left. The beaker is kept on a horizontal table rotating with angular speed ω . This makes the water surface curved so that the difference in the height of water level at the center and at the circumference of the beaker is h ($h \ll H, h \ll r$), as shown in the figure on the right. Take this surface to be approximately spherical with a radius of curvature R . Which of the following is/are correct? (g is the acceleration due to gravity)



(A) $R = \frac{h^2 + r^2}{2h}$

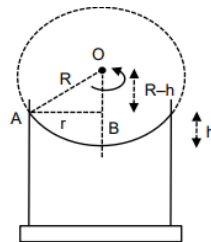
(B) $R = \frac{3r^2}{2h}$

(C) Apparent depth of the bottom of the beaker is close to $\frac{3H}{2} \left(1 + \frac{\omega^2 H}{2g} \right)^{-1}$

(D) Apparent depth of the bottom of the beaker is close to $\frac{3H}{4} \left(1 + \frac{\omega^2 H}{4g} \right)^{-1}$

Answer Key (A, D)

Sol:



In $\triangle OAB$

$$R^2 = (R - h)^2 + r^2$$

$$R^2 = R^2 - 2hR + h^2 + r^2 \Rightarrow 2hR = h^2 + r^2$$

$$\Rightarrow R = \frac{h^2 + r^2}{2h}$$

Now considering equation of surface

$$y = y_0 - \frac{\omega^2 r^2}{2g}$$

$$h = \frac{\omega^2 r^2}{2g}$$

$$\text{Now using : } \frac{H_2}{v} - \frac{H_1}{u} = \frac{H_2 - H_1}{R}$$

$$\Rightarrow \frac{1}{v} + \frac{4}{3(H-h)} = \frac{1-4/3}{-R}$$

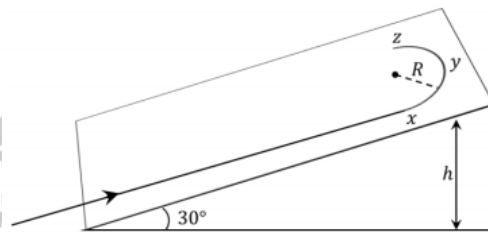
$$\Rightarrow \frac{1}{v} = \frac{1}{3R} - \frac{4}{3H}$$

$$\Rightarrow \frac{1}{v} = \frac{2h}{3r^2} - \frac{4}{3H}$$

$$\Rightarrow \frac{1}{v} = \frac{4}{3H} \left(1 - \frac{\omega^2 H}{4g} \right)$$

$$\Rightarrow v = \frac{3H}{4} \left(1 + \frac{\omega^2 H}{4g} \right)^{-1}$$

8. A student skates up a ramp that makes an angle 30° with the horizontal. He/she starts (as shown in the figure) at the bottom of the ramp with speed v_0 and wants to turn around over a semicircular path xyz of radius R during which he/she reaches a maximum height h (at point y) from the ground as shown in the figure. Assume that the energy loss is negligible and the force required for this turn at the highest point is provided by his/her weight only. Then (g is the acceleration due to gravity)



(A) $v_0^2 - 2gh = \frac{1}{2} gR$

(B) $v_0^2 - 2gh = \frac{\sqrt{3}}{2} gR$

(C) the centripetal force required at points x and z is zero

(D) the centripetal force required is maximum at points x and z

Answer Key (A, D)

Sol:

By the energy conservation (ME) between bottom point and point Y

$$\frac{1}{2} m v_0^2 = mgh + \frac{1}{2} m v_1^2$$

$$\therefore v_1^2 = v_0^2 - 2gh \dots (i)$$

Now at point Y the centripetal force provided by the component of mg

$$\therefore mg \sin 30^\circ = \frac{m v_1^2}{R}$$

$$\therefore v_1^2 = \frac{m v_1^2}{R}$$

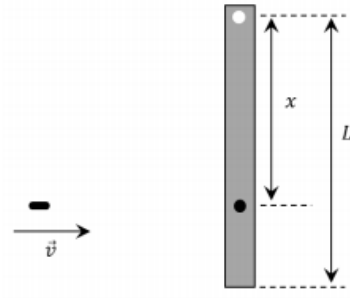
\therefore from (i)

$$\frac{gR}{2} = v_0^2 - 2gh$$

At point x and z of circular path, the points are at same height but less than h . So the velocity more than a point y .

So required centripetal force = $\frac{m v^2}{r}$ is more.

9. A rod of mass m and length L , pivoted at one of its ends, is hanging vertically. A bullet of the same mass moving at speed v strikes the rod horizontally at a distance x from its pivoted end and gets embedded in it. The combined system now rotates with angular speed ω about the pivot. The maximum angular speed ω_M is achieved for $x = x_M$. Then



(A) $\omega = \frac{3vx}{L^2 + 3x^2}$

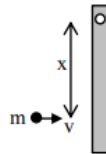
(B) $\omega = \frac{12vx}{L^2 + 12x^2}$

(C) $x_M = \frac{1}{\sqrt{3}}$

(D) $\omega = \frac{v}{2L} \sqrt{3}$

Answer Key (A, C, D)

Sol:



By the angular momentum conservation about the suspension point.

$$mvx = \left(\frac{ml^2}{3} + mx^2 \right) \omega$$

$$\therefore \omega = \frac{mvx}{\frac{ml^2}{3} + mx^2} = \frac{2vx}{l^2 + 3x}$$

For maximum $\omega \Rightarrow \frac{d\omega}{dx} = 0$

$$\therefore x_M = \frac{1}{\sqrt{3}}$$

$$\text{So the } \omega = \frac{v}{2l} \sqrt{3}$$

10. In an X-ray tube, electrons emitted from a filament (cathode) carrying current I hit a target (anode) at a distance d from the cathode. The target is kept at a potential V higher than the cathode resulting in emission of continuous and characteristic X-rays. If the filament current I is decreased to $\frac{1}{2}$, the potential difference V is increased to $2V$, and the separation distance d is reduced to $\frac{d}{2}$, then

- (A) the cut-off wavelength will reduce to half, and the wavelengths of the characteristic X-rays will remain the same
 (B) the cut-off wavelength as well as the wavelengths of the characteristic X-rays will remain the same
 (C) the cut-off wavelength will reduce to half, and the intensities of all the X-rays will decrease
 (D) the cut-off wavelength will become two times larger, and the intensity of all the X-rays will decrease

Answer Key (A, C)

Sol:

$$\lambda_{\min} = \frac{hc}{eV}$$

$$\Rightarrow \lambda_{\min} \propto \frac{1}{V} \Rightarrow (\lambda_{\min})_{\text{new}} = \frac{\lambda_{\min}}{2}$$

$$I = \frac{dN}{dt} \times \frac{hc}{\lambda}$$

$\frac{dN}{dt}$ decreases

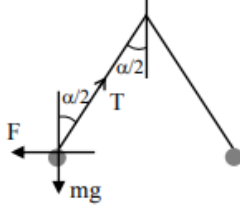
Hence I decreases

11. Two identical non-conducting solid spheres of same mass and charge are suspended in air from a common point by two non-conducting, massless strings of same length. At equilibrium, the angle between the strings is α . The spheres are now immersed in a dielectric liquid of density 800 kg m^{-3} and dielectric constant 21. If the angle between the strings remains the same after the immersion, then
- (A) electric force between the spheres remains unchanged
 (B) electric force between the spheres reduces
 (C) mass density of the spheres is 840 kg m^{-3}
 (D) the tension in the strings holding the spheres remains unchanged

Answer Key (A, C)

Sol:

The net electric force on any sphere is lesser but by Coulomb law the force due to one sphere to another remain the same.



In equilibrium

$$T \cos \frac{\alpha}{2} = mg$$

$$\text{and } T \sin \frac{\alpha}{2} = F$$

After immersed in dielectric liquid.

As given no change in angle α .

$$\text{So } T \cos \frac{\alpha}{2} = mg \sqrt{\rho \rho_0}$$

$$\text{when } \rho = 800 \text{ Kg/m}^3$$

$$\text{and } T \sin \frac{\alpha}{2} = \frac{F}{\epsilon_r}$$

$$\therefore \frac{mg}{F} = \frac{mg - V \rho_0 g}{\frac{F}{\epsilon_r}}$$

$$\frac{1}{\epsilon_r} = 1 - \frac{\rho}{d}$$

d = density of sphere

$$\frac{1}{21} = 1 - \frac{800}{d}$$

$$d = 840$$

12. Starting at time $t = 0$ from the origin with speed 1 ms^{-1} , a particle follows a two-dimensional trajectory in the

x - y plane so that its coordinates are related by the equation $y = \frac{x^2}{2}$. The x and y components of its

acceleration are denoted by a_x and a_y , respectively. Then

(A) $a_x = 1 \text{ ms}^{-2}$ implies that when the particle is at the origin, $a_y = 1 \text{ ms}^{-2}$

(B) $a_x = 0$ implies $a_y = 1 \text{ ms}^{-2}$ at all times

(C) at $t = 0$, the particle's velocity points in the x -direction

(D) $a_x = 0$ implies that at $t = 1 \text{ s}$, the angle between the particle's velocity and the x axis is 45°

Answer Key (A, B, C, D)

Sol:

$$y = \frac{x^2}{2}$$

$$\text{at } t = 0, x = 0, y = 0$$

$$u = 1$$

} given

$$y = \frac{x^2}{2}$$

$$\frac{dy}{dt} = \frac{1}{2} \cdot 2x \frac{dx}{dt}$$

$$\Rightarrow v_y = x v_x$$

Different wrt time

$$a_y = \frac{dx}{dt} \cdot v_x + x a_x$$

$$a_y = v_x^2 + x a_x$$

Option

(A) If $a_x = 1$ and particle is at origin

($x = 0, y = 0$)

$$a_y = v_x^2$$

$$a_y = 1^2 = 1$$

At origin, at $t = 0$ sec

speed = 1 given

(B) Option

$$a_y = v_x^2 + x a_x$$

given in option B, $a_x = 0$

$\Rightarrow a_y = v_x^2$ If $a_x = 0, v_x = \text{constant} = 1$, (all the time)

$\Rightarrow a_y = 1^2 = 1$ (all the time)

(C) at $t = 0, x = 0, v_y = x v_x$

speed = 1

$$v_y = 0, v_x = 1$$

(D) $a_y = v_x^2 + x a_x$

$$v_y = x v_x$$

$a_x = 0$ (given in D option)

$$\Rightarrow a_y = v_x^2$$

If $a_x = 0 \Rightarrow v_x = \text{constant initially } (v_x = 1)$

$$\Rightarrow a_y = 1^2 = 1$$

at $t = 1$ sec $v_y = 0 + a_y \times t = 1 \times 1 = 1$

$$\tan \theta = \frac{v_y}{v_x} = 1$$

($\theta \rightarrow$ angle with x axis)

$$\tan \theta = \frac{v_y}{v_x} = \frac{1}{1} = 1$$

$$\theta = 45^\circ$$

SECTION 3 (Maximum Marks 24)

- This section contains **SIX** (06) questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate / round-off** the value of **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks	:	+4	If ONLY the correct numerical value is entered.
Zero Marks	:	0	In all other cases.

13. A spherical bubble inside water has radius R . Take the pressure inside the bubble and the water pressure to be p_0 . The bubble now gets compressed radially in an adiabatic manner so that its radius becomes $(R - a)$. For $a \ll R$ the magnitude of the work done in the process is given by $(4\pi p_0 R a^2)X$, where X is a constant and $\gamma = C_p/C_v = 41/30$. The value of X is _____.

Answer Key (1.00)

Sol:

$$W = (\Delta P)_{\text{avg}} \times 4\pi R^2 a$$

$$\approx \left| \frac{dP}{2} \cdot 4\pi R^2 a \right|$$

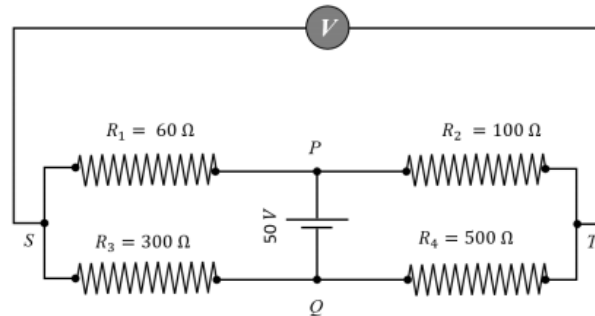
{for small change $(\Delta P)_{\text{avg}}$ arithmetic mean}

$$= P V^\gamma = c \Rightarrow dP = -\gamma \frac{P}{V} dV = \frac{\gamma P_0}{V} dV = -\frac{\gamma P_0}{V} 4\pi R^2 a$$

$$= \frac{\gamma P_0}{2V} \times 4\pi R^2 \times a \times 4\pi R^2 a = \frac{\gamma P_0}{2 \times 4\pi R^3} 4\pi R^2 a \times 4\pi R^2 a = (4\pi p_0 \times a^2) \frac{3\gamma}{2}$$

$$\therefore X = 2.05$$

14. In the balanced condition, the values of the resistances of the four arms of a Wheatstone bridge are shown in the figure below. The resistance R_3 has temperature coefficient $0.0004 \text{ } ^\circ\text{C}^{-1}$. If the temperature of R_3 is increased by $100 \text{ } ^\circ\text{C}$, the voltage developed between S and T will be _____ volt.



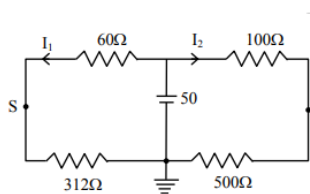
Answer Key (0.27)

Sol:

$$R_3 = 300 (1 + \alpha \Delta T)$$

$$= 312 \text{ } \Omega$$

Now



$$I_1 = \frac{50}{372} \text{ and } I_2 = \frac{50}{600}$$

$$V_S - V_T = 312I_1 - 500I_2$$

$$= 41.94 - 41.67$$

$$= 0.27 \text{ V}$$

15. Two capacitors with capacitance values $C_1 = 2000 \pm 10 \text{ pF}$ and $C_2 = 3000 \pm 15 \text{ pF}$ are connected in series. The voltage applied across this combination is $V = 5.00 \pm 0.02 \text{ V}$. The percentage error in the calculation of the energy stored in this combination of capacitors is _____.

Answer Key (1.30)

Sol:

$$U = \frac{1}{2} \frac{C_1 C_2}{C_1 + C_2} V^2$$

$$\text{Let } C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$$

$$\frac{1}{C_{eq} \pm \Delta C_{eq}} = \frac{1}{C_1 \pm \Delta C_1} + \frac{1}{C_2 \pm \Delta C_2}$$

$$\Rightarrow C_{eq} \pm \Delta C_{eq} \approx \frac{C_1 C_2 + C_1 \Delta C_2 + C_2 \Delta C_1}{C_1 + C_2 + \Delta C_1 + \Delta C_2} = \frac{1200 \left(1 \pm \frac{12}{1200}\right)}{\left(1 \pm \frac{25}{1000}\right)} = 1200 \left| 1 \pm \frac{1}{100} - \frac{1}{200} \right|$$

$$\frac{\Delta U}{U} \times 100 = \frac{\Delta C_{eq}}{C_{eq}} \times 100 + \frac{2\Delta V}{V} \times 100$$

$$\frac{1}{200} \times 100 + 2 \times \frac{0.02}{5} \times 100 = 1.3\%$$

16. A cubical solid aluminium (bulk modulus $= -V \frac{dp}{dv} = 70 \text{ GPa}$) block has an edge length of 1 m on the surface of the earth. It is kept on the floor of a 5 km deep ocean. Taking the average density of water and the acceleration due to gravity to be 10^3 kg m^{-3} and 10 ms^{-2} , respectively, the change in the edge length of the block in mm is _____.

Answer Key (0.24)

Sol:

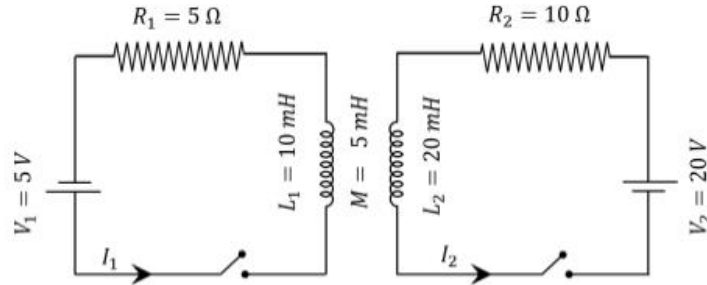
$$\frac{dV}{V} = \frac{3da}{a}$$

$$B = -V \frac{dP}{dV} = \frac{V(\rho gh)}{dV} = \frac{-\rho gh}{3da} a$$

$$70 \times 10^9 = \frac{1 \times 5000 \times 10^{-3} \times 10 \times 1}{3 \times da}$$

$$da = \Delta a = \frac{5}{21} \times 10^{-2} \text{m} = 2.38 \text{mm}$$

17. The inductors of two LR circuits are placed next to each other, as shown in the figure. The values of the self-inductance of the inductors, resistances, mutual-inductance and applied voltages are specified in the given circuit. After both the switches are closed simultaneously, the total work done by the batteries against the induced EMF in the inductors by the time the currents reach their steady state values is _____ mJ.



Answer Key (55.00)

Sol:

Mutual inductance is producing flux in same direction as self inductance.

$$\therefore U = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$$

$$U = \frac{1}{2} \times (10 \times 10^{-3})^2 + \frac{1}{2} \times (20 \times 10^{-3}) \times 2^2 + (5 \times 10^{-3}) \times 1 \times 2 = 55 \text{ MJ}$$

18. A container with 1 kg of water in it is kept in sunlight, which causes the water to get warmer than the surroundings. The average energy per unit time per unit area received due to the sunlight is 700 Wm^{-2} and it is absorbed by the water over an effective area of 0.05 m^2 . Assuming that the heat loss from the water to the surroundings is governed by Newton's law of cooling, the difference (in $^{\circ}\text{C}$) in the temperature of water and the surroundings after a long time will be _____. (Ignore effect of the container, and take constant for Newton's law of cooling = 0.001 s^{-1} , Heat capacity of water = $4200 \text{ J kg}^{-1} \text{ K}^{-1}$)

Answer Key (8.33)

Sol:

$$\frac{dQ}{dt} = e\sigma(T^4 - T_0^4) \dots\dots(i)$$

$$\frac{dQ}{Adt} = e\sigma(T_0 - \Delta T)^4 - T_0^4 = \sigma T_0^4 \left[\left(1 + \frac{\Delta T}{T_0}\right)^4 - 1 \right] = e\sigma T_0^4 \left[\left(1 + 4 \frac{\Delta T}{T_0}\right) - 1 \right]$$

$$\frac{dQ}{Adt} = \sigma e T_0^3 \cdot 4\Delta T \dots\dots(ii)$$

Now from equ. (i)

$$ms \frac{dT}{dt} = \sigma e T (T^4 - T_0^4)$$

$$\frac{dT}{dt} = \frac{\sigma e A}{ms} [(T_0 + \Delta T)^4 - T_0^4] = \frac{\sigma e A}{ms} T_0^4 \times \left[\left(1 + \frac{\Delta T}{T_0}\right)^4 - 1 \right]$$

$$\frac{dT}{dt} = e\Delta T; \left(K = \frac{4\sigma e A T_0^3}{ms} \right)$$

$$\Rightarrow 4\sigma e A T_0^3 = \frac{K}{A} (ms)$$

from equ. (i)

$$\frac{dQ}{Adt} = \sigma e T_0^3 \cdot 4\Delta T$$

$$700 = (K/A) (ms) \Delta T$$

$$\therefore \Delta T = \frac{700 \times 5 \times 10^{-2}}{10^{-3} \times 4200} = \frac{50}{6} = \frac{25}{3}$$

$$\Delta T = 8.33$$

[PART B - CHEMISTRY]

SECTION 1 (Maximum Marks 18)

- This section contains **SIX** (06) questions
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, **BOTH INCLUSIVE**.
- For each question, either the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme.
Full Marks : +3 If **ONLY** the correct integer is entered.
Zero Marks : 0 If the questions is unanswered;
Negative Marks : -1 In all other cases.

1. The 1st, 2nd, and the 3rd ionization enthalpies, I_1 , I_2 , and I_3 , of four atoms with atomic numbers n , $n + 1$, $n + 2$, and $n + 3$, where $n < 10$, are tabulated below. What is the value of n ?

Atomic number	Ionization Enthalpy		
	I_1	I_2	I_3
N	1681	3374	6050
$n + 1$	2081	3952	6122
$n + 2$	496	4562	6910
$n + 3$	739	1451	7733

Answer Key (9)

Sol:

For the species with atomic number $(n + 2)$

$I_2 \gg I_1$ indicating that it has only one valence electron. Hence it is an alkali metal

For the species with atomic number $(n + 3)$

$I_3 \gg I_2$ indicating that it has two valences electrons. Hence it is an alkaline earth metal

\therefore Species with atomic number n should be halogen

given that $n < 10$

\therefore it is fluorine and $n = 9$

2. Consider the following compound in the liquid form:
 O_2 , HF, H_2O , NH_3 , H_2O_2 , CCl_4 , $CHCl_3$, C_6H_6 , C_6H_5Cl
When a charged comb is brought near their flowing stream, how many of them show deflection as per the following figure?



Answer Key (6)

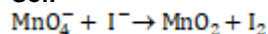
Sol:

Polar compounds show deflection towards a charged object. In the given set HF, H_2O , NH_3 , H_2O_2 , $CHCl_3$ and C_6H_5Cl are polar

3. In the chemical reaction between stoichiometric quantities of $KMnO_4$ and KI in weakly basic solution, what is the number of moles of I_2 released for 4 moles of $KMnO_4$ consumed?

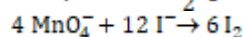
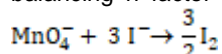
Answer Key (6)

Sol:



$n=3$ $n=1$

balancing 'n' factor



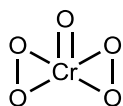
4. An acidified solution of potassium chromate was layered with an equal volume of amyl alcohol. When it was shaken after the addition of 1 mL of 3% H_2O_2 , a blue alcohol layer was obtained. The blue color is due to the formation of a chromium (VI) compound 'X'. What is the number of oxygen atom bonded to chromium through only single bonds in a molecule of 'X'?

Answer Key (4)

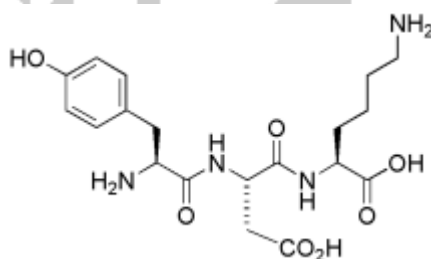
Sol:

Potassium chromate in acidic medium reacts with H_2O_2 to form chromium peroxide CrO_5 , that dissolve in alcohol layer to form a blue colour.

The structure of CrO_5 is



5. The structure of a peptide is given below



If the absolute values of the net charge of the peptide at $\text{pH} = 2$, $\text{pH} = 6$ and $\text{pH} = 11$ are $|z_1|$, $|z_2|$ and $|z_3|$, respectively, then what is $|z_1| + |z_2| + |z_3|$?

Answer Key (5)

Sol:

At $\text{pH} = 2$ (acidic)

The free amino groups get protonated

So 2 $-\text{NH}_2$ groups (of Tyrosine and Lysine) turns to NH_3^+

$$\therefore |z_1| = 2$$

At $\text{pH} = 6$ (nearly neutral)

The peptide form a Zwitter ion (dipolar ion)

\therefore net charge is zero

$$|z_2| = 3$$

At $\text{pH} = 11$ (basic)

The free carboxylic groups of Lysine and glutamic acid and OH group of Tyrosine form anion

$$|z_3| = 3$$

$$|z_1| + |z_2| + |z_3| = 5$$

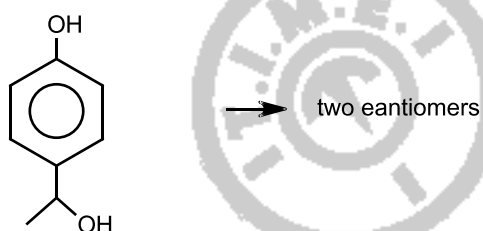
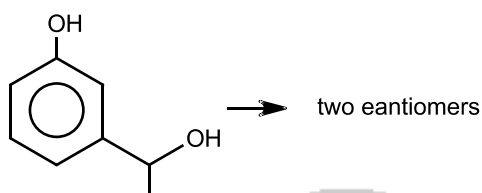
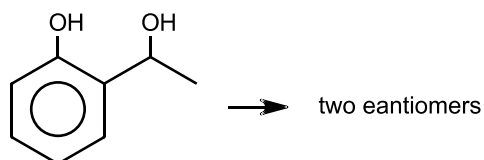
6. An organic compound ($C_8H_{10}O_2$) rotates plane-polarized light. It produces pink color with neutral $FeCl_3$ solution. What is the number of all the possible isomers for this compound?

Answer Key (6)

Sol:

Since the compound ($C_8H_{10}O_2$) answer neutral $FeCl_3$ test it should be phenol
It also rotates the plane polarized light so it should carry chiral carbon

∴ The possible isomers are

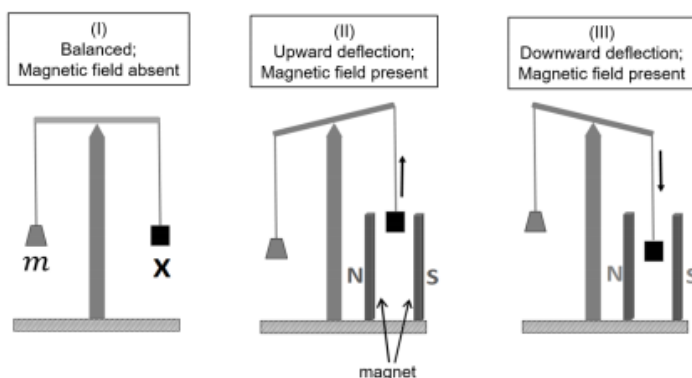


Total isomers = 6

SECTION 2 (Maximum Marks 24)

- This section contains **SIX** (06) questions
 - Each question **FOUR** options. **ONE OR MORE THAN** of these four option(s) is (are) correct answer(s).
 - For each question, choose the option(s) corresponding to (all) the correct answer(s).
 - Answer to each question will be evaluated according to the following marking scheme.
- | | | | |
|-----------------------|---|----|---|
| <i>Full Marks</i> | : | +4 | If only (all) the correct option(s) is (are) chosen: |
| <i>Partial Marks</i> | : | +3 | If all the four options are correct but ONLY three options are chosen; |
| <i>Partial Marks</i> | : | +2 | If three or more options are correct but ONLY two options are chosen, both of which are correct; |
| <i>Partial Marks</i> | : | +1 | If two or more options are correct but ONLY one option is chosen and it is a correct option; |
| <i>Zero Marks</i> | : | 0 | If none of the options is chosen (i.e., the questions is unanswered); |
| <i>Negative Marks</i> | : | -2 | In all other cases. |

7. In an experiment, m grams of a compound X (gas/liquid/solid) taken in a container is loaded in a balance as shown in figure I below. In the presence of a magnetic field, the pan with X is either deflected upwards (figure II), or deflected downwards (figure III), depending on the compound X. Identify the correct statement(s).



- (A) If X is $\text{H}_2\text{O}_{(l)}$, deflection of the pan is upwards
 (B) If X is $\text{K}_4[\text{Fe}(\text{CN})_6]_{(s)}$, deflection of the pan is upwards
 (C) If X is $\text{O}_2_{(g)}$, deflection of the pan is downwards
 (D) If X is $\text{C}_6\text{H}_6_{(l)}$, deflection of the pan is downwards

Answer Key (A, B, C)

Sol:

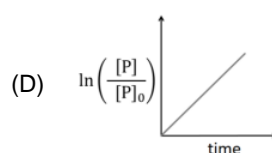
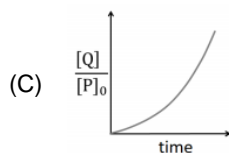
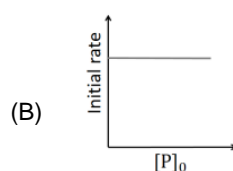
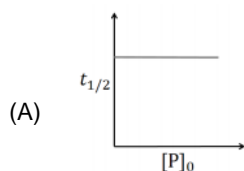
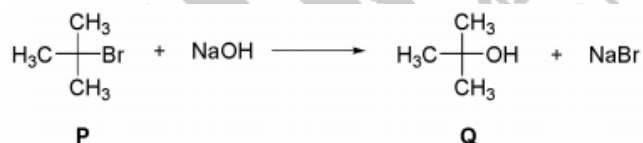
In case of paramagnetic species, the pan deflected downwards as they are attracted by magnetic field.

In the case of diamagnetic species, the pan deflected upwards as they are slightly repelled by magnetic field.

O_2 is paramagnetic while H_2O , C_6H_6 and $\text{K}_4[\text{Fe}(\text{CN})_6]$ are diamagnetic

NOTE: In $\text{K}_4[\text{Fe}(\text{CN})_6]$ the 3d subshell of Fe^{3+} ($3d^6$) in strong ligand field splits as $t_{2g}^6 e_g^0$. So the number of unpaired electron is zero.

8. Which of the following plots is(are) correct for the given reaction? ($[\text{P}]_0$ is the initial concentration of P)



Answer Key (A)

Sol:

The given reaction follows $\text{S}_{\text{N}}1$ kinetics

- (A) For a first order reaction $t_{1/2}$ is independent of initial pressure

\therefore the graph is correct

(B) For a first order reaction initial rate depends on initial pressure
∴ the graph is wrong

(D) According to first order kinetics

$$\ln \frac{P}{P_0} = -kt$$

∴ the plot between $\ln \frac{P}{P_0}$ against time should give a straight line with negative slope.

(C) $\frac{P}{P_0} = e^{-kt}$

$$\frac{Q}{P_0} = \frac{P_0 - P}{P_0} = 1 - \frac{P}{P_0} = 1 - e^{-kt}$$

∴ the curve should have a Y intercept

9. Which among the following statement(s) is(are) true for the extraction of aluminium from bauxite?

(A) Hydrated Al_2O_3 precipitates, when CO_2 is bubbled through a solution of sodium aluminate.

(B) Addition of Na_3AlF_6 lowers the melting point of alumina

(C) CO_2 is evolved at the anode during electrolysis

(D) The cathode is a steel vessel with a lining of carbon

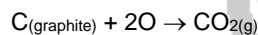
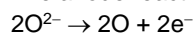
Answer Key (A, B, C, D)

Sol:

During the concentration of bauxite by leaching process hydrated alumina from the solution is precipitated by passing CO_2

In Hall-Heroult's process, Cryolite (Na_3AlF_6) is added to alumina for lowering its melting point.

The anode reaction in Hall-Heroult process is



10. Choose the correct statement(s) among the following.

(A) $\text{SnCl}_2 \cdot 2\text{H}_2\text{O}$ is a reducing agent.

(B) SnO_2 reacts with KOH to form $\text{K}_2[\text{Sn}(\text{OH})_6]$.

(C) A solution of PbCl_2 in HCl contains Pb^{2+} and Cl^- ions

(D) The reaction of Pb_3O_4 with hot dilute nitric acid to give PbO_2 is a redox reaction.

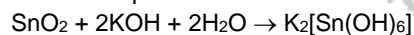
Answer Key (A, B)

Sol:

Due to the strong tendency of Stannous ion (Sn^{2+}) to get oxidized to stannic ion (Sn^{4+})

$\text{SnCl}_2 \cdot 2\text{H}_2\text{O}$ acts as a reducing agent

SnO_2 is amphoteric and reacts with strong base like KOH



Pb^{2+} ions form insoluble PbCl_2 in HCl

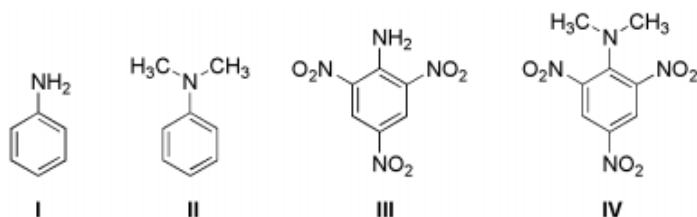


Pb_3O_4 is a mixed oxide with a stoichiometric formula



There is no change in oxidation state during the reaction

11. Consider the following four compounds I, II, III, and IV.



Choose the correct statement(s).

- (A) The order of basicity is **II > I > III > IV**
 (B) The magnitude of pK_b difference between I and II is more than that between **III** and **IV**.
 (C) Resonance effect is more in **III** than in **IV**
 (D) Steric effect makes compound **IV** more basic than **III**

Answer Key (C, D)

Sol:

The least basic is compound III (2,4,6-trinitro aniline) due to the maximum electron withdrawing effect of NO_2 ($-\text{R}$ group)

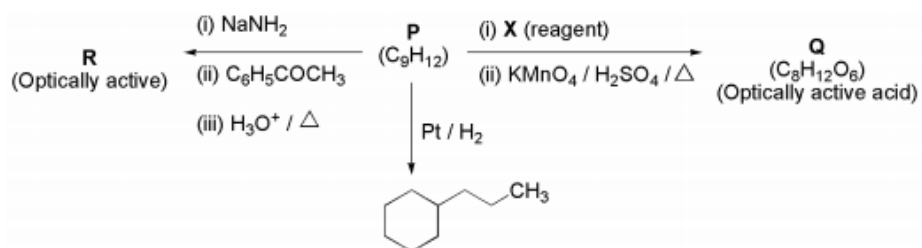
In compound IV (N,N-dimethyl-2,4,6-trinitro aniline) the amino group get pushed out of the plane of benzene ring due to steric effect so the magnitude of delocalization of lone pair of nitrogen is much lowered and hence make the lone pair more available for protonation

There is a large difference in pK_b value of compound III and IV

Compound II is most basic

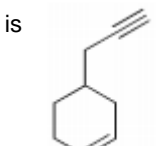
Compound IV is less basic than compound II due to $-\text{I}$ effect of NO_2 group

12. Consider the following transformations of a compound **P**.

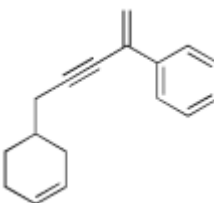


Choose the correct option(s)

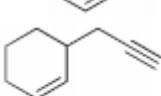
(A) **P** is



(B) **X** is Pd-C/quinoline/ H_2



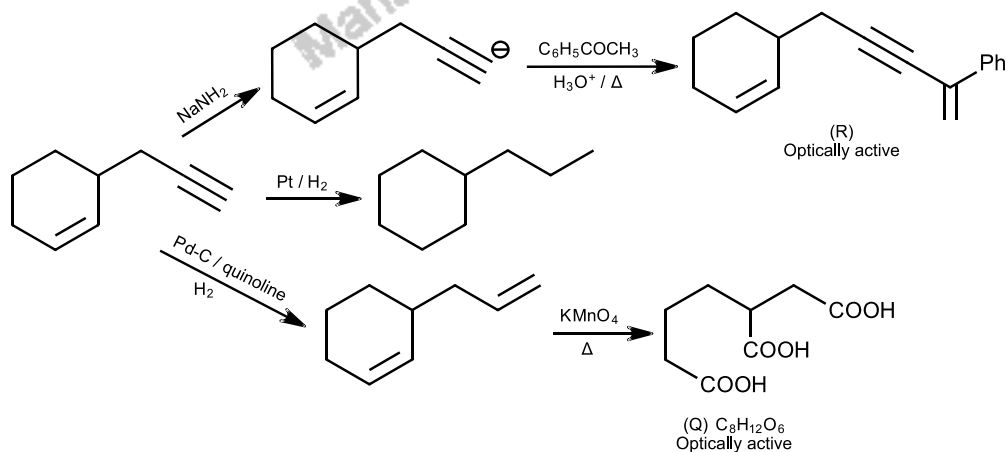
(C)



(D)

Answer Key (B, C)

Sol:



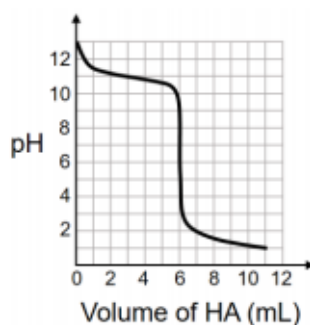
SECTION 3 (Maximum Marks 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate / round-off** the value of **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks : +4 If **ONLY** the correct numerical value is entered.

Zero Marks : 0 In all other cases.

13. A solution of 0.1 M weak base (B) is titrated with 0.1 M of a strong acid (HA). The variation of pH of the solution with the volume of HA added is shown in the figure below. What is the pK_b of the base? The neutralization reaction is given by $B + HA \rightarrow BH^+ + A^-$



Answer Key (3.00)

Sol:

From the graph,

Volume of HA used at the equivalence point = 6 mL

i.e., 6 mL of the base is taken for titration

On adding 3 mL of the acid, it forms a basic buffer for which

$$pOH = pK_b + \log \frac{[\text{Salt}]}{[\text{Base}]}$$

and $[\text{Salt}] = [\text{Base}]$

$$\therefore pOH = pK_b$$

On adding 3 mL HA, $pH = 11$

$$\therefore pOH = 14 - 11 = 3$$

14. Liquids **A** and **B** form ideal solution for all compositions of **A** and **B** at 25 °C. Two such solutions with 0.25 and 0.50 mole fractions of **A** have the total vapor pressures of 0.3 and 0.4 bar, respectively. What is the vapor pressure of pure liquid **B** in bar?

Answer Key (0.2)

Sol:

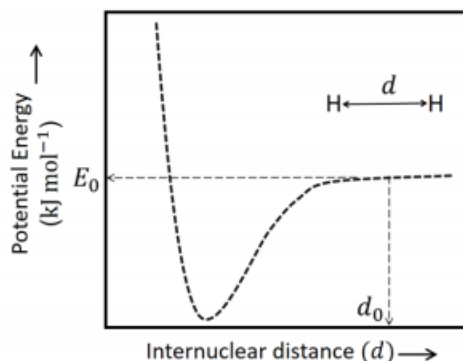
$$P_A^0 X_A + P_B^0 X_B = P_T$$

$$0.25 P_A^0 + 0.75 P_B^0 = 0.3 \text{ --- (1)}$$

$$0.50 P_A^0 + 0.50 P_B^0 = 0.4 \text{ --- (2)}$$

$$(1) \times 2 - (2) \Rightarrow P_B^0 = 0.2$$

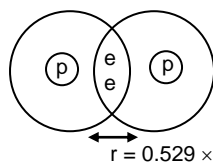
15. The figure below is the plot of potential energy versus internuclear distance (d) of H_2 molecule in the electronic ground state. What is the value of the net potential energy E_0 (as indicated in the figure) in kJ mol^{-1} , for $d = d_0$ at which the electron-electron repulsion and the nucleus-nucleus repulsion energies are absent? As reference, the potential energy of H atom is taken as zero when its electron and the nucleus are infinitely far apart. Use Avogadro constant as $6.023 \times 10^{23} \text{ mol}^{-1}$



Answer Key (5257.00)

Sol:

At $d = d_0$ all the repulsion in H_2 are absent

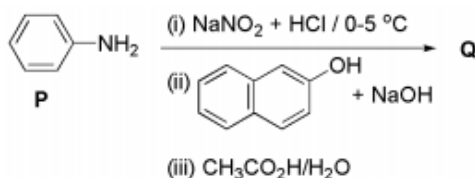


$$\text{Potential energy of a H atom } k \frac{e^2}{r} = \frac{9 \times 10^9 \times (1.602 \times 10^{-19})^2}{0.529 \times 10^{-10}}$$

$$\text{Potential energy of a } H_2 \text{ molecule} = \frac{2 \times 9 \times 10^9 \times (1.602 \times 10^{-19})^2}{0.529 \times 10^{-10}}$$

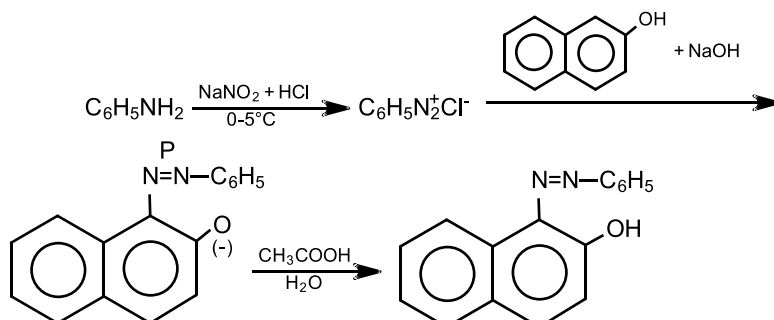
$$\text{For one mole} = \frac{2 \times 9 \times 10^9 \times (1.602 \times 10^{-19})^2}{0.529 \times 10^{-10}} \times 6.023 \times 10^{23} = 5259.00$$

16. Consider the reaction sequence from **P** to **Q** shown below. The overall yield of the major product **Q** from **P** is 75%. What is the amount in grams of **Q** obtained from 9.3 mL of **P**? (Use density of **P** = 1.00 g mL^{-1} ; Molar mass of C = 12.0, H = 1.0, O = 16.0 and N = 14.0 g mol^{-1})



Answer Key (18.60)

Sol:



Molecular weight of P (aniline) = 93 g mol⁻¹

Given that density of P is 1 gm mL⁻¹

9.3 mL of P weighs 9.3 g

∴ no. of moles of P = 0.1

0.1 mole of P should give 0.1 mole Q if the conversion is 100%

Here the yield is only 75%

∴ no. of moles of Q formed = 0.075

Molecular weight of Q (C₁₆H₁₂N₂O) = 248 g

amount of Q formed = 248 × 0.075 = 18.6 g

17. Tin is obtained from cassiterite by reduction with coke. Use the data given below to determine the minimum temperature (in K) at which the reduction of cassiterite by coke would take place.

At 298 K: $\Delta_f H^\circ$ (SnO_{2(s)}) = -581.0 kJ mol⁻¹, $\Delta_f H^\circ$ (CO_{2(g)}) = -394.0 kJ mol⁻¹,

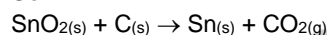
S° (SnO_{2(s)}) = 56.0 J K⁻¹ mol⁻¹, S° (Sn_(s)) = 52.0 J K⁻¹ mol⁻¹,

S° (C_(s)) = 6.0 J K⁻¹ mol⁻¹, S° (CO_{2(g)}) = 210.0 J K⁻¹ mol⁻¹.

Assume that the enthalpies and the entropies are temperature independent.

Answer Key (935.00)

Sol:



$$\Delta S^\circ = [52 + 210] - [56 + 6] = 200 \text{ J K}^{-1} \text{ mol}^{-1}$$

$$\Delta H^\circ = -394 + 581 = 187 \text{ kJ mol}^{-1}$$

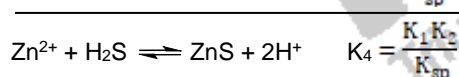
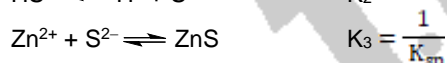
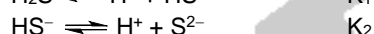
At equilibrium $\Delta H = T\Delta S$

$$T = \frac{187 \times 10^3}{200} = 935 \text{ K}$$

18. An acidified solution of 0.05 M Zn²⁺ is saturated with 0.1 M H₂S. What is the minimum molar concentration (M) of H⁺ required to prevent the precipitation of ZnS? Use K_{sp} (ZnS) = 1.25 × 10⁻²² and overall dissociation constant of H₂S, $K_{NET} = K_1K_2 = 1 \times 10^{-21}$.

Answer Key (0.20)

Sol:



$$K_4 = \frac{1 \times 10^{-21}}{1.25 \times 10^{-22}}$$

$$K_4 = \frac{[\text{H}^+]^2}{[\text{Zn}^{2+}][\text{H}_2\text{S}]}$$

$$[\text{H}^+]^2 = \frac{10}{1.25} \times 0.05 \times 0.1 = 4 \times 10^{-2}$$

$$[\text{H}^+] = 0.2 \text{ M}$$

[PART C - MATHEMATICS]

SECTION 1 (Maximum Marks 18)

- This section contains **SIX** (06) questions
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, **BOTH INCLUSIVE**.
- For each question, either the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme.
Full Marks : +3 If ONLY the correct integer is entered.
Zero Marks : 0 If the questions is unanswered;
Negative Marks : -1 In all other cases.

1. For a complex number z , let $\text{Re}(z)$ denote the real part of z . Let S be the set of all complex numbers z satisfying $z^4 - |z|^4 = 4iz^2$, where $i = \sqrt{-1}$. Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with $\text{Re}(z_1) > 0$ and $\text{Re}(z_2) < 0$, is _____.

Answer key (8)

Sol:

Let $z = x + iy$, $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

Given $z^4 - |z|^4 = 4iz^2$

$\therefore |z|^2 = z \cdot \bar{z}$

$z^4 - \bar{z}^2 z^2 = 4iz^2$

$z^2 - \bar{z}^2 = 4i$

$(z + \bar{z})(z - \bar{z}) = 4i$

$2x \times 2iy = 4i$

$xy = 1 \rightarrow (A)$

Also, given $\left. \begin{matrix} \text{Re}(z_1) > 0 \Rightarrow x_1 > 0 \Rightarrow y_1 > 0 \\ \text{Re}(z_2) < 0 \Rightarrow x_2 < 0 \Rightarrow y_2 < 0 \end{matrix} \right\} \rightarrow (1)$

$$\begin{aligned} |z_1 - z_2|^2 &= |(x_1 - x_2) + (y_1 - y_2)i|^2 \\ &= \left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \right)^2 \\ &= x_1^2 - x_1x_2 - x_2x_1 + x_2^2 + y_1^2 - y_1y_2 - y_2y_1 + y_2^2 \rightarrow (2) \end{aligned}$$

Using (1) each terms of RHS of $|z_1 - z_2|^2$ is +ve

AM \geq GM

$$\Rightarrow \frac{x_1^2 + (-x_1x_2) + (x_2x_1) + \dots + y_2^2}{8} \geq [x_1^2 \times (-x_1x_2) \times \dots \times y_2^2]$$

$$x_1^2 + (-x_1x_2) + \dots + y_2^2 \geq 8(1)^{1/8} \text{ using (A)}$$

$$(1) \Rightarrow \text{min imum of } |z_1 - z_2|^2 = 8$$

2. The probability that a missile hits a target successfully is 0.75. In order to destroy the target completely, at least three successful hits are required. Then the minimum number of missiles that have to be fired so that the probability of completely destroying the target is **NOT** less than 0.95, is _____

Answer key (6)

Sol:

Let x be the no: of n missiles fired

and p = probability that a missile hits a target successfully

$$= \frac{3}{4}$$

$$q = 1 - p = \frac{1}{4} \text{ (Binomial)}$$

$$\begin{aligned} \text{Given } P(\text{hits atleast three}) &= 1 - [P(X=0) + P(X=1) + P(X=2)] \\ &\geq \frac{95}{100} \end{aligned}$$

$$\Rightarrow 1 - \left[{}^n C_0 \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^n + {}^n C_1 \left(\frac{3}{4}\right)^1 \left(\frac{1}{4}\right)^{n-1} + {}^n C_2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^{n-2} \right] \geq \frac{95}{100}$$

$$\frac{5}{100} \geq \frac{1}{4^n} + \frac{3n}{4^n} + \frac{9n(n-1)}{2 \times 4^n}$$

$$2 \times 4^n \times \frac{1}{20} \geq 2 + 6n + 9n^2 - 9n$$

$$\frac{2^{2n-1}}{5} \geq 2 - 3n + 9n^2$$

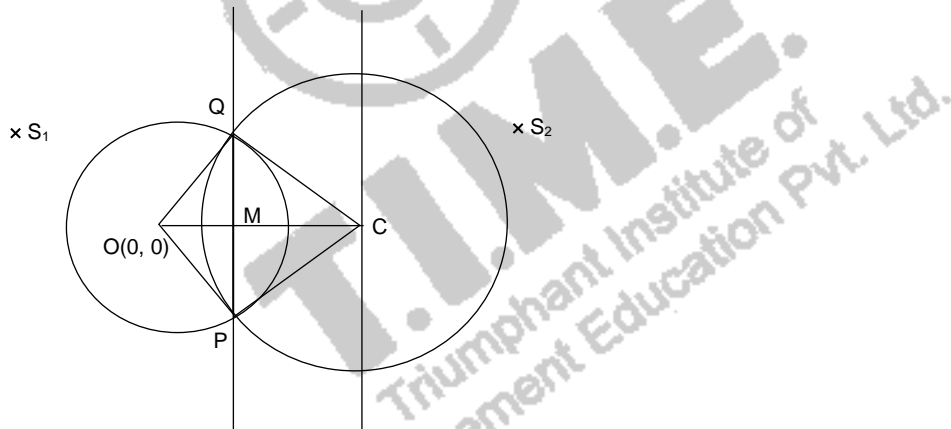
substitute $n = 1, 2, \dots$ we get ,

$$n = 6$$

3. Let O be the centre of the $x^2 + y^2 = r^2$, where $r > \frac{\sqrt{5}}{2}$. Suppose PQ is a chord of this circle and the equation of the line passing through P and Q is $2x + 4y = 5$. If the centre of the circumcircle of the triangle OPQ lies on the line $x + 2y = 4$, then the value of r is _____.

Answer key (2)

Sol:



Let $O(0, 0)$ be the centre & $x^2 + y^2 = r^2$, (S_1) and PQ be the chord on this circle (S_1)

Equation of PQ , $2x + 4y = 5 \rightarrow (1)$ (Given)

Let S_2 be the circumcircle of $\triangle OPQ$. Let C be the centre & S_2

Given C lie on the line

$$x + 2y = 4 \rightarrow (2)$$

Let M be the point where OC intersects PQ

$$OM = \perp^r \text{ distance from } (0, 0) \text{ to } (1) = \frac{|5|}{\sqrt{4+16}} = \frac{5}{\sqrt{20}}$$

Also, $OC = QC$ (radius & same circle)

$$\text{From } \triangle OQM, OQ = r \text{ and } QM^2 = r^2 - OM^2 \rightarrow (3)$$

$$\text{From } \triangle QMC, QM^2 = QC^2 - MC^2 \rightarrow (4)$$

$$\text{From } (3) \text{ \& } (4) \Rightarrow r^2 = QC^2 - MC^2 + CM^2$$

S_{mee} $OC \perp^r$ to line (2)

$$\overline{OC} = \perp^r \text{ distance from } (0, 0) \text{ to } (2)$$

$$OC = \frac{4}{\sqrt{5}} (= QC)$$

$$\text{Also, } MC = OC - OM$$

$$r^2 = QC^2 - MC^2 + OM^2$$

$$= \frac{4}{\sqrt{5}} - \frac{5}{\sqrt{20}}$$

$$= \left(\frac{4}{\sqrt{5}}\right)^2 - \left(\frac{3}{\sqrt{20}}\right)^2 + \left(\frac{5}{\sqrt{20}}\right)^2$$

$$= \frac{3}{\sqrt{20}}$$

$$= 4$$

$$r = 2$$

4. The trace of a square matrix is defined to be the sum of its diagonal entries. If A is a 2×2 matrix such that the trace of A is 3 and the trace of A^3 is -18 , then the value of the determinant of A is _____.

Answer key (5)

Sol:

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ Given trace of } A = a + d = 3$$

$$d = (3 - a)$$

$$\therefore A = \begin{bmatrix} a & b \\ c & (3-a) \end{bmatrix}$$

$$\begin{aligned} \text{Consider } A^2 &= \begin{bmatrix} a & b \\ c & (3-a) \end{bmatrix} \begin{bmatrix} a & b \\ c & (3-a) \end{bmatrix} \\ &= \begin{bmatrix} a^2 + c & ab + 3b - ab \\ ac + 3c - ac & bc + (3-a)^2 \end{bmatrix} \\ &= \begin{bmatrix} a^2 + bc & 3b \\ 3c & bc + (3-a)^2 \end{bmatrix} \end{aligned}$$

$$A^3 = \begin{bmatrix} a & b \\ c & (3-a) \end{bmatrix} \times \begin{bmatrix} a^2 + bc & 3b \\ 3c & bc + (3-a)^2 \end{bmatrix}$$

$$\text{Trace of } (A^3) = a^3 + abc + 3bc + 3bc + (3-a)bc + (3-a)^2$$

$$-18 = a^3 + abc + 3bc + 3bc + 3bc - abc + (3-a)^3$$

$$-18 = a^3 + 9bc + 27 - 27a + 9a^2 - a^3$$

$$-18 = 9[bc + 3 - 3a + 3a^2]$$

$$-2 = 3a^2 - 3a + bc + 3$$

$$3a^2 - 3a + bc = -5$$

$$\text{Det. } A, |A| = a(3-a) - bc$$

$$= 3a - 3a^2 - bc$$

$$= 5$$

5. Let the functions $f : (-1, 1) \rightarrow \mathbb{R}$ and $g : (-1, 1) \rightarrow (-1, 1)$ be defined by $f(x) = |2x - 1| + |2x + 1|$ and $g(x) = x - [x]$, where $[x]$ denotes the greatest integer less than or equal to x . Let $f \circ g : (-1, 1) \rightarrow \mathbb{R}$ be the composite function defined by $(f \circ g)(x) = f(g(x))$. Suppose c is the number of points in the interval $(-1, 1)$ at which $f \circ g$ is **NOT** continuous and suppose d is the number of points in the interval $(-1, 1)$ at which $f \circ g$ is **NOT** differentiable. Then the value of $c + d$ is _____.

Answer key (4)

Sol:

$$f(x) = |2x - 1| + |2x + 1|; g(x) = x - [x] = \{x\} \text{ where } -1 < x < 1$$

Case i: $-1 < x < \frac{-1}{2}$

$$\Rightarrow \{x\} < \frac{1}{2} \left[\begin{array}{l} \text{for example } x = -0.6 \\ = -1 + 0.4 \\ \{x\} = 0.4 < \frac{1}{2} \end{array} \right]$$

$$(\text{fog})(x) = |2\{x\} - 1| + |2\{x\} + 1|$$

$$= -2\{x\} + 1 + 2\{x\} + 1 \left[\begin{array}{l} \Rightarrow 0 < \{x\} < \frac{1}{2} \\ 0 < 2\{x\} < 1 \\ \Rightarrow 2\{x\} - 1 < 0 \\ \Rightarrow |2\{x\} - 1| = -2\{x\} + 1 \end{array} \right]$$

$$= 2$$

Case ii: $\frac{-1}{2} < x < 0$

$$\Rightarrow 0.5 < \{x\} < 1 \left[\begin{array}{l} \text{for example } x = -0.1 \\ = -1 + 0.7 \\ \{x\} = 0.7 \end{array} \right]$$

$$\therefore (\text{fog})(x) = 2\{x\} - 1 + 2\{x\} + 1$$

$$\begin{aligned} &= 4\{x\} \\ &= 4[x - \{x\}] \\ &= 4(x + 1) \end{aligned}$$

$$y = 4x + 4$$

$$-4x + y = 4$$

$$\frac{x}{-1} + \frac{y}{4} = 1$$

Case: iii: $0 < x < \frac{1}{2}$

$$\therefore \{x\} = x$$

$$\therefore 0 < \{x\} < \frac{1}{2}$$

$$0 < 2\{x\} < 1$$

$$\begin{aligned} (\text{fog})(x) &= |2\{x\} - 1| + |2\{x\} + 1| \\ &= -2\{x\} + 1 + 2\{x\} + 1 \\ &= 2 \end{aligned}$$

Case iv: $\frac{1}{2} < x < 1$

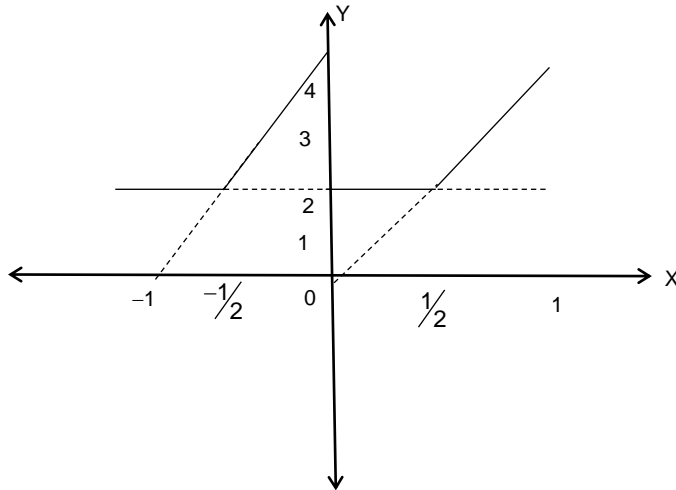
$$\therefore \{x\} = x$$

$$0.5 < \{x\} < 1$$

$$1 < 2\{x\} < 2$$

$$\begin{aligned} \therefore (\text{fog})(x) &= |2\{x\} - 1| + |2\{x\} + 1| \\ &= 2\{x\} - 1 + 2\{x\} + 1 \\ &= 4\{x\} \\ &= 4[x - \{x\}] \\ &= 4[x - 0] = 4x \end{aligned}$$





From the diagram $C =$ root of points where fog is real root contains $= 1$ (at $x = 0$)

Also $d =$ no of points where fog is not differentiable $= 3 \left[\text{at } x = \frac{-1}{2}, C, \frac{1}{2} \right]$

$$\therefore c + d = 1 + 3 = 4$$

6. The value of the limit $\lim_{x \rightarrow \frac{\pi}{2}} \frac{4\sqrt{2}(\sin 3x + \sin x)}{\left(2 \sin 2x \sin \frac{5x}{2}\right) - \left(\sqrt{2} + \sqrt{2} \cos 2x + \cos \frac{3x}{2}\right)}$ is _____.

Answer key (8)

Sol:

$$\begin{aligned} & \lim_{x \rightarrow \frac{\pi}{2}} \frac{4\sqrt{2}(\sin 2x + \sin x)}{\left(2 \sin 2x \sin \frac{3x}{2}\right) + \cos \frac{5x}{2} - \left(\sqrt{2} + \sqrt{2} \cos 2x + \cos \frac{3x}{2}\right)} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{4\sqrt{2} \times 2 \sin 2x \cos x}{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{7x}{2}\right) - \sqrt{2}(1 + \cos 2x) + \cos \frac{5x}{2} - \cos \frac{3x}{2}} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{8\sqrt{2} \times (2 \sin x \cos^2 x)}{\left(\cos \frac{x}{2} - \cos \frac{3x}{2}\right) + \left(\cos \frac{5x}{2} - \cos \frac{7x}{2}\right) - \sqrt{2} \cdot 2 \cos^2 x} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{16\sqrt{2} \sin x \cos^2 x}{8 \sin \frac{x}{2} \times (2 \sin x \cos x) \cos x - 2\sqrt{2} \cos^2 x} \end{aligned}$$

$$\text{cancelling } \cos^2 x = \lim_{x \rightarrow \frac{\pi}{2}} \frac{16\sqrt{2} \sin x}{(8) \sin \frac{x}{2} \sin x - 2\sqrt{2}}$$

$$= \frac{16\sqrt{2} \sin\left(\frac{\pi}{2}\right)}{(8) \sin\left(\frac{\pi}{4}\right) \times \sin \frac{\pi}{2} - 2\sqrt{2}} = \frac{16\sqrt{2}}{4\sqrt{2} - 2\sqrt{2}} = 8$$

SECTION 2 (Maximum Marks 24)

- This section contains **SIX** (06) questions
- Each question **FOUR** options. **ONE OR MORE THAN** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme.

<i>Full Marks</i>	:	+4	If only (all) the correct option(s) is (are) chosen:
<i>Partial Marks</i>	:	+3	If all the four options are correct but ONLY three options are chosen;
<i>Partial Marks</i>	:	+2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
<i>Partial Marks</i>	:	+1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
<i>Zero Marks</i>	:	0	If none of the options is chosen (i.e., the question is unanswered);
<i>Negative Marks</i>	:	-2	In all other cases.

7. Let b be a nonzero real number, Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function such that $f(0) = 1$. If the derivative f' of f satisfies the equation $f'(x) = \frac{f(x)}{b^2 + x^2}$ for all $x \in \mathbb{R}$, then which of the following statements

is/are **TRUE**?

- (A) If $b > 0$, then f is an increasing function (B) If $b < 0$, then f is a decreasing function
 (C) $f(x) f(-x) = 1$ for all $x \in \mathbb{R}$ (D) $f(x) - f(-x) = 0$ for all $x \in \mathbb{R}$

Answer key (A, C)

Sol:

$$f'(x) = \frac{f(x)}{b^2 + x^2} \Rightarrow \frac{f'(x)}{f(x)} = \frac{1}{b^2 + x^2}$$

$$\Rightarrow \log f(x) = \frac{1}{b} \tan^{-1}\left(\frac{x}{b}\right) + C$$

$$\therefore f(0) = 1$$

$$\Rightarrow C = 0$$

$$\log f(x) = \frac{1}{b} \tan^{-1} \frac{x}{b}$$

$$f(x) = e^{\frac{1}{b} \tan^{-1}\left(\frac{x}{b}\right)}$$

$$\therefore f'(x) = e^{\frac{1}{b} \tan^{-1}\left(\frac{x}{b}\right)} \cdot \frac{1}{b^2 + x^2} > 0 \quad \forall x \in \mathbb{R}$$

$f(x)$ is increasing

$$f(x) \cdot f(-x) = e^{\frac{1}{b} \tan^{-1}\left(\frac{x}{b}\right)} \cdot e^{-\frac{1}{b} \tan^{-1}\left(\frac{x}{b}\right)} = 1$$

8. Let a and b be positive real numbers such that $a > 1$ and $b < a$. Let P be a point in the first quadrant that lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Suppose the tangent to the hyperbola at P passes through the point $(1, 0)$

and suppose the normal to the hyperbola at P cuts off equal intercepts on the coordinate axes. Let Δ denote the area of the triangle formed by the tangent at P , the normal at P and the x -axis. If e denotes the eccentricity of the hyperbola, then which of the following statements is/are **TRUE**?

- (A) $1 < e < \sqrt{2}$ (B) $\sqrt{2} < e < 2$ (C) $\Delta = a^4$ (D) $\Delta = b^4$

Answer key (A, D)

Sol:

Tangent at P

$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1 \text{ passes through } (1, 0)$$

$$\sec \theta = a$$

Now slope of AP = 1

$$\frac{b \sec \theta}{a \tan \theta} = 1 \Rightarrow b = \tan \theta$$

$$\text{Now } b^2 = a^2 (e^2 - 1)$$

$$\tan^2 \theta = \sec^2 \theta (e^2 - 1)$$

$$e^2 - 1 = \sin^2 \theta$$

$$e^2 - 1 \in [0, 1]$$

$$e^2 \in [1, 2]$$

$$\Rightarrow 1 < e < \sqrt{2}$$

Now A(1, 0) and P(a sec θ , b tan θ) = (sec² θ , tan² θ)

$$AP = \sqrt{\tan^2 \theta + \tan^2 \theta} = \sqrt{2} \tan \theta$$

$$\begin{aligned} \text{Area} &= \frac{1}{2} (AP)^2 = \frac{1}{2} \times 2 \tan^4 \theta \\ &= b^4 \end{aligned}$$

9. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be functions satisfying $f(x+y) = f(x) + f(y) + f(x)f(y)$ and $f(x) = xg(x)$ for all $x, y \in \mathbb{R}$. If $\lim_{x \rightarrow 0} g(x) = 1$, then which of the following statements is/are **TRUE**?

- (A) f is differentiable at every $x \in \mathbb{R}$
(C) The derivative $f'(1)$ is equal to 1

- (B) If $g(0) = 1$, then g is differentiable at every $x \in \mathbb{R}$
(D) The derivative $f'(0)$ is equal to 1

Answer key (A, B, D)

Sol:

Put $x = y = 0$ in given relation

$$\Rightarrow f(0) = f(0) + f(0) + f^2(0)$$

$$\Rightarrow f(0) = 0 \quad (0) - 1$$

$$f(x+y) = f(x) + f(y) + f(x) \cdot f(y)$$

$$\lim_{y \rightarrow 0} \frac{f(x+y) - f(x)}{y} = \lim_{y \rightarrow 0} \frac{f(y)[1 + f(x)]}{y}$$

$$\Rightarrow f'(x) = 1 + f(x)$$

$$\Rightarrow f'(0) = 1 + f(0)$$

$$\Rightarrow f'(0) = 1 + 0$$

$$\Rightarrow f'(0) = 1$$

$$\therefore \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} \frac{f(x)}{x} = 1$$

$$\text{Again } \frac{f'(x)}{1 + f(x)} = 1$$

$$\Rightarrow \int \frac{f'(x)}{1 + f(x)} dx = \int dx$$

$$\Rightarrow \log(1 + f(x)) = x + c$$

$$\Rightarrow \log(1 + f(x)) = x \quad [\because \text{at } x = 0 \Rightarrow c = 0]$$

$$\Rightarrow 1 + f(x) = e^x$$

$$\Rightarrow f(x) = e^x - 1$$

$$\Rightarrow f'(x) = e^x$$

$$\Rightarrow f'(1) = e \text{ \& } f'(0) = e^0 = 1$$

$$g(x) = \frac{f(x)}{x} = \frac{e^x - 1}{x}$$

$$g'(0^+) = \lim_{x \rightarrow 0} \frac{g(c+h) - g(c)}{h}$$

$$\text{If } g(0) = 1$$

$$\Rightarrow g'(0^+) = \lim_{h \rightarrow 0} \frac{\frac{e^h - 1}{h} - 1}{h} = \lim_{h \rightarrow 0} \frac{e^h - 1 - h}{h^2} = \frac{1}{2}$$

$$g'(0^-) = \lim_{h \rightarrow 0} \frac{g(0-h) - g(0)}{-h} = \lim_{h \rightarrow 0} \frac{e^{-h} - 1 + h}{h^2} = \frac{1}{2}$$

$g(x)$ is differentiable

10. Let $\alpha, \beta, \gamma, \delta$ be real numbers such that $\alpha^2 + \beta^2 + \gamma^2 \neq 0$ and $\alpha + \gamma = 1$. Suppose the point $(3, 2, -1)$ is the mirror of the point $(1, 0, -1)$ with respect to the plane $\alpha x + \beta y + \gamma z = \delta$. Then which of the following statements is/are TRUE?

(A) $\alpha + \beta = 2$

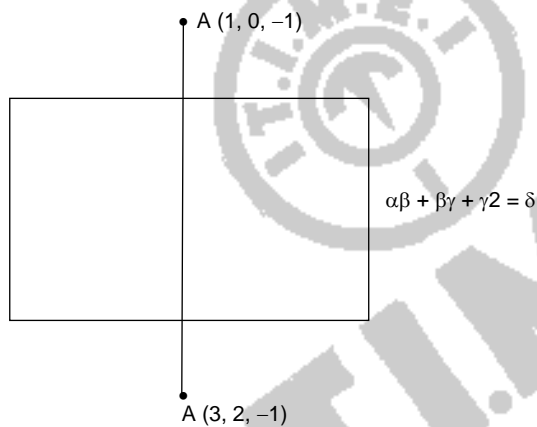
(B) $\delta - \gamma = 3$

(C) $\delta + \beta = 4$

(D) $\alpha + \beta + \gamma = \delta$

Answer key (A, B, C)

Sol:



Find point at $AA' = B(2, 1, -1)$ [$B = \mu P$ of AA']

$$2\alpha + \beta - \gamma = \delta \rightarrow (1)$$

$$\alpha + \gamma = 1$$

dir's of AA' $(2, 2, 0)$

$$\frac{\alpha}{2} = \frac{\beta}{2} = \frac{\gamma}{0} = \lambda$$

$$2\lambda + 0 = 1, \alpha = 2\lambda, \beta = 2\lambda, \gamma = 0$$

$$\lambda = \frac{1}{2}$$

$$\therefore \alpha = 1, \beta = 1, \gamma = 0, \delta = 3$$

Now (D) $= 2 \neq \delta$

$$(A) = 1 + 1 = 2$$

$$= 0 + 3 = 3$$

$$\delta + \gamma = 3 - 0 = 3$$

$$(A) \alpha + \beta = 1 + 1 = 2$$

$$(B) \delta - \gamma = 3 - 0 = 3$$

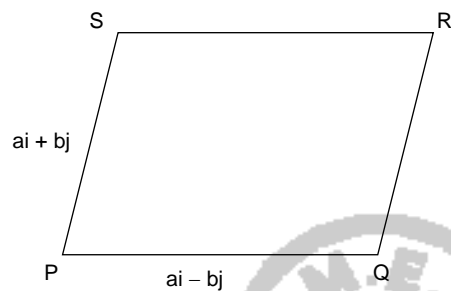
$$(C) \delta + \beta = 1 + 3 = 4$$

$$(D) \alpha + \beta + \gamma = 2 \neq \delta$$

11. Let a and b be positive real numbers, Suppose $\vec{PQ} = a\hat{i} + b\hat{j}$ and $\vec{PS} = a\hat{i} - b\hat{j}$ are adjacent sides of a parallelogram PQRS. Let \vec{u} and \vec{v} be the projection vectors of $\vec{w} = \hat{i} + \hat{j}$ along \vec{PQ} and \vec{PS} , respectively. If $|\vec{u}| + |\vec{v}| = |\vec{w}|$ and if the area of the parallelogram PQRS is 8, then which of the following statements is/are TRUE?
- (A) $a + b = 4$
 (B) $a - b = 2$
 (C) The length of the diagonal PR of the parallelogram PQRS is 4
 (D) \vec{w} is an angle bisector of the vectors \vec{PQ} and \vec{PS}

Answer key (A, C)

Sol:



$$\vec{u} = \left\{ (\hat{i} + \hat{j}) \cdot \frac{a\hat{i} + b\hat{j}}{\sqrt{a^2 + b^2}} \right\} \frac{a\hat{i} + b\hat{j}}{\sqrt{a^2 + b^2}} = \left(\frac{a+b}{\sqrt{a^2 + b^2}} \right) \cdot \frac{a\hat{i} + b\hat{j}}{\sqrt{a^2 + b^2}}$$

$$\vec{v} = \left\{ (\hat{i} + \hat{j}) \cdot \frac{a\hat{i} - b\hat{j}}{\sqrt{a^2 + b^2}} \right\} \frac{a\hat{i} - b\hat{j}}{\sqrt{a^2 + b^2}} = \left(\frac{a-b}{\sqrt{a^2 + b^2}} \right) \frac{a\hat{i} - b\hat{j}}{\sqrt{a^2 + b^2}}$$

Given $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & 0 \\ a & -b & 0 \end{vmatrix} = 8$

$$\Rightarrow ab = 4$$

Given $|\vec{u}| + |\vec{v}| = |\vec{w}|$

$$\frac{a+b}{\sqrt{a^2 + b^2}} + \frac{a-b}{\sqrt{a^2 + b^2}} = \sqrt{2}$$

a > b

$$2a = \sqrt{2}\sqrt{a^2 + b^2}$$

$$2a^2 = a^2 + b^2$$

$$a + b = 4$$

$$a - b = 0$$

$$\text{diagonal} = \vec{PS} + \vec{PQ} = 2a\hat{i}$$

$$\text{length} = 2a = 4$$

a < b

$$2a = \sqrt{2}\sqrt{a^2 + b^2}$$

$$a^2 = b^2$$

$$a = b = 2$$

12. For nonnegative integers s and r , let $\binom{s}{r} = \begin{cases} \frac{s!}{r!(s-r)!} & \text{if } r \leq s, \\ 0 & \text{if } r > s \end{cases}$

For positive integers m and n , let $g(m, n) = \sum_{p=0}^{m+n} \frac{f(m, n, p)}{n+p}$

where for any nonnegative integer p , $f(m, n, p) = \sum_{i=0}^p \binom{m}{i} \binom{n+i}{p} \binom{p+n}{p-i}$.

Then which of the following statements is/are TRUE?

- (A) $g(m, n) = g(n, m)$ for all positive integers m, n
 (B) $g(m, n+1) = g(m+1, n)$ for all positive integers m, n
 (C) $g(2m, 2n) = 2g(m, n)$ for all positive integers m, n
 (D) $g(2m, 2n) = (g(m, n))^2$ for all positive integers m, n

Answer key (A, B, D)

Sol:

$$\begin{aligned} & \frac{m!}{(m-i)!i!} \times \frac{(n+i)!}{(n+i-p)!} \times \frac{(p+n)!}{(p-i)(n+i)!} \\ &= \frac{m!}{(m-i)!} \frac{(p+n)!}{p!n!} \frac{n!}{(n+i-p)!} \frac{1}{(p-i)!} \\ &= \frac{m!}{(m-i)!} {}^{p+n}C_p {}^nC_{p-i} \\ &= {}^{p+n}C_p \sum_{i=0}^p {}^mC_i {}^nC_{p-i} \\ &= {}^{p+n}C_p \left[{}^mC_0 {}^nC_p + {}^mC_1 {}^nC_{p-1} + {}^mC_2 {}^nC_{p-2} + \dots + {}^mC_p {}^nC_0 \right] \\ (1+x)^m (1+x)^n &= \left[{}^mC_0 + {}^mC_1x + {}^mC_2x^2 + \dots \right] \left[{}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_px^p + \dots + {}^{p+n}C_p {}^{m+n}C_p + \dots \right] \\ g(m, n) &= \sum_{p=0}^{m+n} {}^{m+n}C_p = 2^{m+n} \\ (C) \quad g(2m, 2n) &= 2^{2m+2n} \\ 2g(m, n) &= 2 \cdot 2^{m+n} \\ g(2m, 2n) &\neq 2g(m, n) \\ (B) \quad g(m, n+1) &= 2^{m+n+1} = g(m+1, n) \\ (A) \quad g(m, n) &= 2^{m+n} = 2^{n+m} = g(n, m) \\ (D) \quad g(2m, 2n) &= 2^{2m+2n} = 2^{2(m+n)} \\ &= [g(m, n)]^2 \end{aligned}$$

SECTION 3 (Maximum Marks 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate / round-off** the value of **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme.
Full Marks : +4 If ONLY the correct numerical value is entered.
Zero Marks : 0 In all other cases.

13. An engineer is required to visit a factory for exactly four days during the first 15 days of every month and it is mandatory that **no** two visits take place on consecutive days. Then the number of all possible ways in which such visits to the factory can be made by the engineer during 1 – 15 June 2021 is _____.

Answer key (495.00)

Sol:

Consider 15 days as 15 balls

In this 11 are green and 4 are red, no red is consecutive

Now first we arrange 11 green balls that is in one way.

Then we have 12 gaps in this you put 4 Red balls in ${}^{12}C_4$ ways

$$\text{So answer is } {}^{12}C_4 = \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = 495.00$$

14. In a hotel, four rooms are available. Six persons are to be accommodated in these four rooms in such a way that each of these rooms contains at least one person and at most two persons. Then the number of all possible ways in which this can be done is _____.

Answer key (1080.00)

Sol:

Select 2 persons from 6 persons in 6C_2 ways and keep in one room.

Then we have 4 persons. Now select 2 persons from 4 persons in 4C_2 ways and keep in second room.

Then we have two persons arrange that two persons in two rooms in 2! Ways

Arrange 4 rooms in $\frac{4!}{2! 2!}$ ways

$$\text{i.e., } {}^6C_2 \cdot {}^4C_2 \cdot {}^2C_1 \cdot {}^1C_1 \cdot \frac{4!}{2! 2!} (\because 1, 1, 2, 2)$$

$$= 15 \cdot 6 \cdot 2 \cdot 1 \cdot 6$$

$$= 1080.00$$

15. Two fair dice, each with faces numbered 1, 2, 3, 4, 5 and 6, are rolled together and the sum of the numbers on the faces is observed. This process is repeated till the sum is either a prime number or a perfect square. Suppose the sum turns out to be a perfect square before it turns out to be a prime number. If p is the probability that this perfect square is an odd number, then the value of 14p is _____.

Answer key (8.00)

Sol:

$$\text{Perfect square } \begin{cases} 4 \rightarrow (1, 3), (3, 1), (2, 2) \\ 9 \rightarrow (6, 3), (3, 6), (5, 4), (4, 5) \end{cases}$$

7 possibilities

$$\text{Prime } \rightarrow 2 \rightarrow (1, 1)$$

$$3 \rightarrow (1, 2), (2, 1)$$

$$5 \rightarrow (1, 4), (2, 3), (3, 2), (4, 1)$$

$$7 \rightarrow (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$$

$$11 \rightarrow (5, 6), (6, 5)$$

15- possibilities

$$P(\text{square (or) prime}) = \frac{15 + 7}{36} = \frac{22}{36}$$

$$P(\text{not a square or prime}) = 1 - \frac{22}{36} = \frac{14}{36}$$

$$P\left[\frac{\text{perfect square is odd}}{\text{perfect square before prime}}\right] = \frac{\frac{4}{36} + \left(\frac{14}{36} \times \frac{4}{36}\right) + \left(\frac{14}{36} \times \frac{14}{36} \times \frac{4}{36}\right) + \dots}{\frac{7}{36} + \left(\frac{14}{36} \times \frac{7}{36}\right) + \left(\frac{14}{36} \times \frac{14}{36} \times \frac{7}{36}\right) + \dots} = \frac{4}{7} = p$$

$$7p = 4 \Rightarrow 14p = 8.00$$

16. Let the function $f: [0, 1] \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{4^x}{4^x + 2}$. Then the value of

$$f\left(\frac{1}{40}\right) + f\left(\frac{2}{40}\right) + f\left(\frac{3}{40}\right) + \dots + f\left(\frac{39}{40}\right) - f\left(\frac{1}{2}\right) \text{ is } \underline{\hspace{2cm}}.$$

Answer key (19.00)

Sol:

$$\begin{aligned} & f\left(\frac{1}{40}\right) + f\left(\frac{2}{40}\right) + \dots + f\left(\frac{19}{40}\right) + f\left(\frac{20}{40}\right) + f\left(\frac{21}{40}\right) + \dots + f\left(\frac{39}{40}\right) - f\left(\frac{1}{2}\right) \\ &= f\left(\frac{1}{40}\right) + f\left(\frac{2}{40}\right) + \dots + f\left(\frac{19}{40}\right) + f\left(\frac{1}{2}\right) + f\left(1 - \frac{19}{40}\right) + \dots + f\left(1 - \frac{1}{40}\right) - f\left(\frac{1}{2}\right) \\ &= \left[f\left(\frac{1}{40}\right) + f\left(1 - \frac{1}{40}\right) \right] + \left[f\left(\frac{2}{40}\right) + f\left(1 - \frac{2}{40}\right) \right] + \dots + \left[f\left(\frac{19}{40}\right) + f\left(1 - \frac{19}{40}\right) \right] \end{aligned}$$

$$\text{Now } f(x) + f(1-x) = \frac{4^x}{4^x + 2} + \frac{4^{1-x}}{4^{1-x} + 2} = \frac{4^x}{4^x + 2} + \frac{4}{4 + 2 \cdot 4^x} = \frac{4^x + 2}{4^x + 2} = 1$$

$$= 1 + 1 + 1 + \dots \text{19 times}$$

$$= 19.00$$

17. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that its derivative f' is continuous and $f(\pi) = -6$.

If $F: [0, \pi] \rightarrow \mathbb{R}$ is defined by $F(x) = \int_0^x f(t) dt$, and if $\int_0^\pi (f'(x) + F(x)) \cos x dx = 2$, then the value of $f(0)$ is _____.

Answer key (4.00)

Sol:

$$F(x) = \int_0^x f(t) dt, \quad f(\pi) = -6$$

$$F'(x) = f(x)$$

$$\int_0^\pi [f'(x) + F(x)] \cos x dx = 2$$

$$\int_0^\pi f'(x) \cos x dx + \int_0^\pi F(x) \cos x dx = 2$$

$$[\cos x f(x)]_0^\pi - \int_0^\pi (-\sin x) f(x) dx + \int_0^\pi F(x) \cos x dx = 2$$

$$\cos \pi \cdot f(\pi) - \cos 0 \cdot f(0) + \int_0^\pi \sin x f(x) dx + \int_0^\pi F(x) \cos x dx = 2$$

$$(-1)(-6) - f(0) + [\sin x F(x)]_0^\pi - \int_0^\pi \cos x F(x) dx + \int_0^\pi F(x) \cos x dx = 6 - f(0) + 0 = 2$$

$$\therefore f(0) = 4.00$$

18. Let the function $f : (0, \pi) \rightarrow \mathbb{R}$ be defined by $f(\theta) = (\sin \theta + \cos \theta)^2 + (\sin \theta - \cos \theta)^4$. Suppose the function f has a local minimum at θ precisely when $\theta \in \{\lambda_1, \pi, \dots, \lambda_r, \pi\}$, where $0 < \lambda_1 < \dots < \lambda_r < \pi$. Then the value of $\lambda_1 + \dots + \lambda_r$ is _____.

Answer key (0.50)

Sol:

$$f(\theta) = (\sin \theta + \cos \theta)^2 + (\sin \theta - \cos \theta)^4$$

$$= 1 + \sin 2\theta + (1 - \sin 2\theta)^2$$

$$= 1 + \sin 2\theta + 1 + \sin^2 2\theta - 2 \sin 2\theta$$

$$= \sin^2 2\theta - \sin 2\theta + 2$$

$$= \left(\sin 2\theta - \frac{1}{2} \right)^2 - \frac{1}{4} + 2$$

$$f(\theta) = \left(\sin 2\theta - \frac{1}{2} \right)^2 + \frac{7}{4}$$

$$\therefore f(\theta) \text{ is minimum at } \sin 2\theta - \frac{1}{2} = 0$$

$$\sin 2\theta = \frac{1}{2}$$

$$2\theta = \left[\frac{\pi}{6}, \frac{5\pi}{6} \right]$$

$$\theta = \left[\frac{\pi}{12}, \frac{5\pi}{12} \right]$$

$$\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$$

$$\therefore \text{sum} = \frac{1}{12} + \frac{5}{12} = \frac{1}{2} = 0.50$$



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