QUESTIONS & ANSWER KEYS FOR JEE (ADVANCED)-2021(PAPER 1)

[MATHEMATICS]

SECTION 1

- This section contains **SIX** (04) questions
- Each question **FOUR** option (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks	:	+3	If ONLY the correct options is chosen;
Zero Marks	:	0	If none of the options is chosen (i.e., the question is unanswered);
Negative Marks	:	–1	In all other cases.

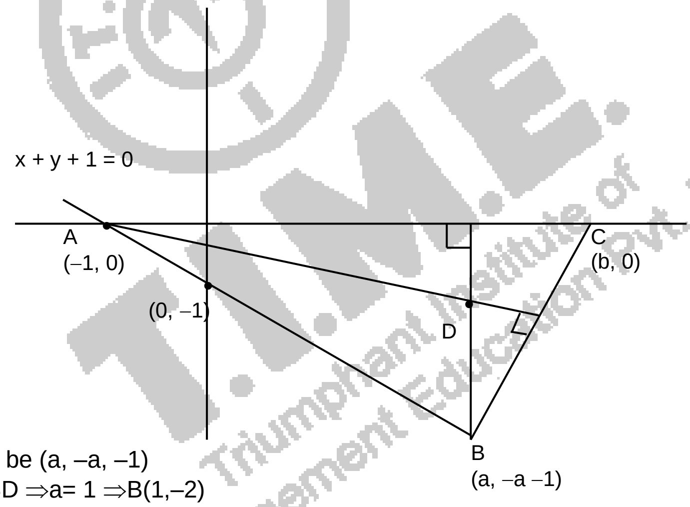
Q.1 Consider a triangle Δ whose two sides lie on the x-axis and the line $x + y + 1 = 0$. If the orthocenter of Δ is $(1, 1)$, then the equation of the circle passing through the vertices of the triangle Δ is

(A) $x^2 + y^2 - 3x + y = 0$ (B) $x^2 + y^2 + x + 3y = 0$

(C) $x^2 + y^2 + 2y - 1 = 0$ (D) $x^2 + y^2 + x + y = 0$

Answer Key (B)

Sol:



Let vertex B be (a, –a, –1)

Line AC $\perp$BD $\Rightarrow$a= 1 $\Rightarrow$B(1,–2)

Line AD $\perp$ BC

$M_1.m_2 = -1$

$\frac{1}{2}.\frac{2}{b-1} = -1 \Rightarrow b = 0$

Centroid of ΔABC is $\left(0, -\frac{2}{3}\right)$

Now G(centroid) divides line joining circum centre (O) and ortho centre (D) in the ratio 1: 2

$2h + 1 = 0$ $2k + 1 = -z$

$h = -\frac{1}{2}$ $k = -\frac{3}{2}$

$\Rightarrow$ circum centre is $\left(-\frac{1}{2}, -\frac{3}{2}\right)$

Equation of circum circle is (passing through C(0,0)) is

$x^2 + y^2 + x + 3y = 0$

Q.2 The area of the region

$$\left\{(x, y) : 0 \le x \le \frac{9}{4}, \quad 0 \le y \le 1, \quad x \ge 3y, \quad x + y \ge 2\right\}$$

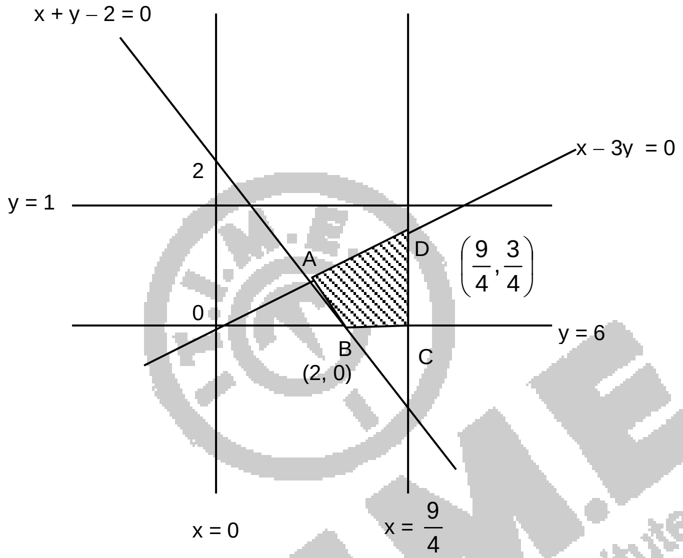
is

(A) $\frac{11}{32}$ (B) $\frac{35}{96}$ (C) $\frac{37}{96}$ (D) $\frac{13}{32}$

Answer Key (A)

Sol: $x + y - 2 = 0$

$A\left(\frac{3}{2}, \frac{1}{2}\right); B(2, 0); C\left(\frac{9}{4}, 0\right); D\left(\frac{9}{4}, \frac{3}{4}\right)$



$$\frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_4 & y_2 - y_4 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -\frac{3}{4} & \frac{1}{2} \\ -\frac{1}{4} & -\frac{3}{4} \end{vmatrix} = \frac{1}{2}\left[\frac{9}{16} + \frac{1}{8}\right] = \frac{1}{2} \times \frac{11}{16} = \frac{11}{32}$$

Q.3 Consider three sets $E_1 = \{1, 2, 3\}$, $F_1 = \{1, 3, 4\}$ and $G_1 = \{2, 3, 4, 5\}$. Two elements are chosen at random, without replacement, from the set E_1, and let S_1 denote the set of these chosen elements. Let $E_2 = E_1 - S_1$ and $F_2 = F_1 \cup S_1$. Now two elements are chosen at random, without replacement, from the set F_2 and let S_2 denote the set of these chosen elements.

Let $G_2 = G_1 \cup S_2$. Finally, two elements are chosen at random, without replacement, from the set G_2 and let S_3 denote the set of these chosen elements.
Let $E_3 = E_2 \cup S_3$. Given that $E_1 = E_3$, let p be the conditional probability of the event $S_1 = \{1, 2\}$. Then the value of p is

(A) $\frac{1}{5}$ (B) $\frac{3}{5}$ (C) $\frac{1}{2}$ (D) $\frac{2}{5}$

Answer Key (A)

Sol: $P = \frac{P(S_1 \cap (E_1 = E_3))}{P(E_1 = E_3)} = \frac{P(B_{12})}{P(B_{12}) + P(B_{13}) + P(B_{23})}$

$P(B_{12}) = \frac{1}{3} \cdot \frac{{}^3C_1}{{}^4C_2} \cdot \frac{1}{{}^5C_2}$

$$P(B_{13}) = \frac{1}{3} \cdot \frac{{}^{2}C_1}{{}^{3}C_2} \cdot \frac{1}{{}^{5}C_2}$$

$$P(B_{23}) = \frac{1}{3}\left[\frac{{}^{3}C_2}{{}^{4}C_2} \cdot \frac{1}{{}^{4}C_2} + \frac{{}^{3}C_1}{{}^{4}C_2} \cdot \frac{1}{{}^{5}C_2}\right]$$

$$\therefore P = = \frac{1}{5}$$

Q.4 Let $\theta_1, \theta_2, \ldots, \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + \cdots + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}$, $z_k = z_{k-1}e^{i\theta_k}$ for $k = 2, 3, \ldots, 10$, where $i = \sqrt{-1}$. Consider the statements P and Q given below:

$$P : |z_2 - z_1| + |z_3 - z_2| + \cdots + |z_{10} - z_9| + |z_1 - z_{10}| \leq 2\pi$$

$$Q : |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + \cdots + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \leq 4\pi$$

Then,

(A) P is **TRUE** and Q is **FALSE**

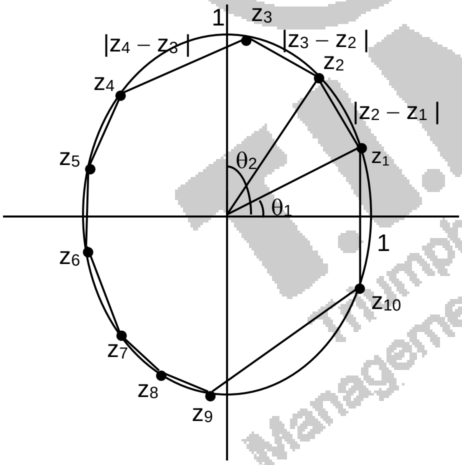
(B) Q is **TRUE** and P is **FALSE**

(C) both P and Q are **TRUE**

(D) both P and Q are **FALSE**

Answer Key (C)

Sol:



$|z_2 - z_1| + |z_3 - z_2| + \ldots |z_1 - z_{10}|$ = sum of all sides of a polygon with 10 sides inscribed in a unit circle $\leq 2\pi(1)$

i.e; $\leq 2\pi$

$\therefore$ P is true

$z_2^2 - z_1^2 = (\cos 2\theta_2 + i\sin 2\theta_2) - (\cos 2\theta_1 + i\sin 2\theta_1)$

$= -2\sin(\theta_2 + \theta_1)\sin(\theta_2 - \theta_1) + i2\cos(\theta_2 + \theta_1)\sin(\theta_2 - \theta_1)$

$= 2\sin(\theta_2 - \theta_1)[-\sin(\theta_2 + \theta_1) + i\cos(\theta_2 + \theta_1)]$

$|z_2^2 - z_1^2| = 2|\sin(\theta_2 - \theta_1)| \times 1 \leq 2|\theta_2 - \theta_1|$ $[\because \sin x \leq x$ if x is in radian$]$

$\therefore |z_2^2 - z_1^2| + |z_3^2 - z_2^2| \ldots + |z_{10}^2 - z_1^2|$

$\leq 2[|\theta_2 - \theta_1| + |\theta_3 - \theta_2| + \ldots\ldots.. + .]$

$\leq 2 \times 2\pi$ (from the figure)

$\leq 4\pi$ (True)

SECTION 2

- This section contains **THREE** (03) questions stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate / round-off** the value to **TWO** decimal places
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks	:	+2	If ONLY the correct numerical value is entered at the designated place;
Zero Marks	:	0	In all other cases.

Question Stem for Question No. 5 and 6

Question Stem

Three numbers are chosen at random, one after another with replacement, from the set $S = \{1,2,3,\ldots,100\}$. Let p_1 be the probability that the maximum of chosen numbers is at least 81 and p_2 be the probability that the minimum of chosen numbers is at most 40.

Q.5 The value of $\frac{625}{4}p_1$ is ___ .

Answer Key (76.25)

Sol: Probability that maximum of chosen numbers is at least 81

1 – probability that maximum of chosen number is at most 80

$$= 1-\frac{80\times80\times80}{100\times100\times100}=1-\frac{64}{125}$$

$$\Rightarrow p_1=\frac{61}{125}$$

$$\frac{625p_1}{4}=\frac{625}{4}\times\frac{61}{125}=\frac{305}{4}=76.25$$

Q.6 The value of $\frac{125}{4}p_2$ is ___ .

Answer Key (24.50)

Sol: Probability that minimum of chosen numbers is at most 40

= 1 – probability that minimum of chosen numbers is at least 41

$$=1-\left(\frac{600}{100}\right)^3$$

$$=1-\frac{27}{125}=\frac{98}{125}\Rightarrow p_2=\frac{98}{125}$$

$$\therefore\frac{125}{4}p_2=24.50$$

Question Stem for Question No. 7 and 8

Question Stem

Let α, β and γ be real numbers such that the system of linear equations

$$x + 2y + 3z = \alpha$$
$$4x + 5y + 6z = \beta$$
$$7x + 8y + 9z = \gamma - 1$$

is consistent. Let $|M|$ represent the determinant of the matrix

$$M = \begin{bmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Let P be the plane containing all those (α, β, γ) for which the above system of linear equations is consistent, and D be the **square** of the distance of the point $(0, 1, 0)$ from the plane P.

Q.7 The value of $|M|$ is __ .

Answer Key (1.00)

Q.8 The value of D is __ .

Answer Key (1.50**)**

Solutions 7 & 8:

Sol: The 3rd equation can be written as A(2nd) + B(3rd) [Θ Infinite number of solution]

$7x + 8y + 9z - (\gamma - 1) = A(4x + 5y + 6z - \beta) + B(x + 2y + 3z - \alpha)$

$4A + B = 7$

$5A + 2B = 8$

$A = 2, B = -1$

Equating Const. term: $-(\gamma - 1) = -A\beta - \alpha B \Rightarrow -(\gamma - 1) \equiv 2\beta + \alpha$

$\alpha - 2\beta + \gamma = 1$

$$M = \begin{vmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \alpha - 2\beta + \gamma = 1$$

Plane P : $x - 2y + z = 1$

Perpendicular distance $= \left|\frac{3}{\sqrt{6}}\right| = P$

$\Rightarrow P^2 = \frac{9}{6} = 1.5$

Question Stem for Question No. 9 and 10

Question Stem

Consider the lines L_1 and L_2 defined by

$$L_1\colon\ x\sqrt{2} + y - 1 = 0 \text{ and } L_2\colon\ x\sqrt{2} - y + 1 = 0$$

For a fixed constant λ, let C be the locus of a point P such that the product of the distance of P from L_1 and the distance of P from L_2 is λ^2. The line $y = 2x + 1$ meets C at two points R and S, where the distance between R and S is $\sqrt{270}$.

Let the perpendicular bisector of RS meet C at two distinct points R' and S'. Let D be the **square** of the distance between R' and S'.

Q.9 The value of λ^2 is ___ .

Answer Key (9.00)

Sol: $\left(\frac{\sqrt{2}x+y-1}{\sqrt{3}}\right)\left(\frac{\sqrt{2}x-y+1}{\sqrt{3}}\right)=\lambda^2$

$\left(\frac{2x^2-(y-1)^2}{3}\right)=\lambda^2$, The Curve is: $\left|2x^2-(y-1)^2\right|=3\lambda^2$ .....(1)

Let $R:(x_1,y_1)$ $S:(x_2,y_2)$ Substituting in $y=2x+1$, $y_1-y_2=2(x_1-x_2)$

$y_1=2x_1+1$ and $y_2=2x_2+1 \Rightarrow (y_1-y_2)=2(x_1-x_2)$

RS = $\sqrt{5(x_1-x_2)^2}=\sqrt{5}\left|x_1-x_2\right|$

Solve the equation (1) and $y=2x+1$ we get

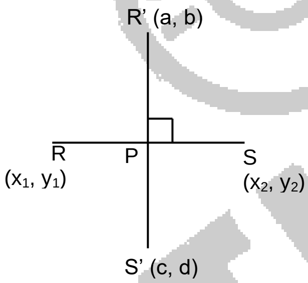
$\left|2x^2-(2x^2)\right|=3\lambda^2 \Rightarrow x^2=\frac{3\lambda^2}{2}$

RS = $\sqrt{30}\lambda=\sqrt{270} \Rightarrow 30\lambda^2=270 \Rightarrow \lambda^2=9$

Q.10 The value of D is ___ .

Answer Key (77.14)

Sol:



$P:\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

Here $x_1+x_2=0$

P : (0, 1)

Equation of R'S' **:** $(y-1)=-\frac{1}{2}(x-0) \Rightarrow x+2y=2$

D = $(a-c)^2+(b-d)^2=5(b-d)^2$

Solving $x+2y=2$ and $\left|2x^2-(y-1)^2\right|=3\lambda^2$

$\Rightarrow (y-1)^2=\left(\frac{\sqrt{3}\lambda}{\sqrt{7}}\right)^2$

$y-1=\pm\frac{\sqrt{3}\lambda}{\sqrt{7}} \Rightarrow b_1=1+\frac{\sqrt{3}\lambda}{\sqrt{7}}, b_2=1-\frac{\sqrt{3}\lambda}{\sqrt{17}}$

D = $5\left(\frac{2\sqrt{3}\lambda}{\sqrt{7}}\right)^2=\frac{5\times4\times27}{7}=77.14$

SECTION 3

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MOR THAN ONE** of these four option(s) is (are) correct answer(s).
- Answer to ach question will be evaluated according to the following marking scheme.

Full Marks:	+4	If only (all) the correct option(s) is (are) chose,
Partial Marks:	+3	If all the four options are correct but ONLY three options are chosen;
Partial Marks:	+2	If three or more options are correct but ONLY two options are chosen, both of which are correct.
Partial Marks:	+1	If two or more options are correct but ONLY one option is chosen and it is a correction option;
Zero Marks	0	If unanswered;
Negative Marks	–2	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
  Choosing ONLY (A), (B) and (D) will get +4 marks;
  Choosing ONLY (A) and (B) will get +2 marks;
  Choosing ONLY (A) and (D) will get +2 marks;
  Choosing ONLY (B) and (D) will get +2 marks;
  Choosing ONLY (A) will get +1 mark;
  Choosing ONLY (B) will get +1 mark;
  Choosing ONLY (D) will get +1 mark;
  Choosing no option(s) (i.e., the question is unanswered) will get 0 mark and
  Choosing any other option(s) will get –2 marks.

Q.11 For any 3×3 matrix M, let $|M|$ denote the determinant of M. Let

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix}$$

If Q is a nonsingular matrix of order 3×3, then which of the following statements is (are) **TRUE** ?

(A) $F = PEP$ and $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $|EQ + PFQ^{-1}| = |EQ| + |PFQ^{-1}|$

(C) $|(EF)^3| > |EF|^2$

(D) Sum of the diagonal entries of $P^{-1}EP + F$ is equal to the sum of diagonal entries of $E + P^{-1}FP$

Answer Key (A, B, D)

Sol: $PEP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{pmatrix}$

$P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(B) Given $|EQ + PFQ^{-1}| = |EQ| + |PFQ^{-1}|$

$|E| = 0$ and $|F| = 0$ $|Q| \neq 0$

$|EQ| = |E||Q| = 0, |PFQ^{-1}| = |P||F||Q^{-1}| = \dfrac{|P||F|}{|Q|} = 0$

$T = EQ + PFQ^{-1}$

$TQ = Q[EQ + PFQ^{-1}] = EQ^2 + PF = EQ^2 + P^2EP = EQ^2 + EP = E(Q^2 + P)$

$|TQ| = |E(Q^2 + P)| \Rightarrow |T|\,|Q| = |E|\,|Q^2 + P| = 0 \Rightarrow |T| = 0$ (as $|Q| \neq 0$)

(C) $\left|(EF)^3\right| > |EF|^2$ which is false as 0 not greater than zero

(D) as $P^2 = I \Rightarrow P^{-1} = P$ so $P^{-1}FP = PFP = P^2EP^2 = I\,E\,I = E$

so, $E + P^{-1}FP = E + E = 2E$ $\quad(\Theta\ I^{-1} = I)$

$P^{-1}EP + F \Rightarrow PEP + F = 2PEP$

$Tr(2PEP) = 2Tr(PEP) = 2Tr(EPP) = 2Tr(E)$

Q.12 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$$

Then which of the following statements is (are) **TRUE** ?

(A) f is decreasing in the interval $(-2, -1)$

(B) f is increasing in the interval $(1, 2)$

(C) f is onto

(D) Range of f is $\left[-\frac{3}{2}, 2\right]$

Answer Key (A, B)

Sol: $f(x) = \dfrac{x^2 - 3x - 6}{x^2 + 2x + 4}$

(Using Qutioent rule)

$f'(x) = \dfrac{5x(x+4)}{(x^2 + 2x + 4)^2}$

$f(-4) = \dfrac{11}{6}, f(0) = -\dfrac{3}{2}, \lim\limits_{x \to \pm\infty} f(x) = 1$

Range: $\left[-\dfrac{3}{2}, \dfrac{11}{6}\right]$, $f(x)$ is into

Q.13 Let E, F and G be three events having probabilities

$P(E) = \frac{1}{8}$, $P(F) = \frac{1}{6}$ and $P(G) = \frac{1}{4}$, and let $P(E \cap F \cap G) = \frac{1}{10}$.

For any event H, if H^c denotes its complement, then which of the following statements is (are) **TRUE** ?

(A) $P(E \cap F \cap G^c) \leq \frac{1}{40}$

(B) $P(E^c \cap F \cap G) \leq \frac{1}{15}$

(C) $P(E \cup F \cup G) \leq \frac{13}{24}$

(D) $P(E^c \cap F^c \cap G^c) \leq \frac{5}{12}$

Answer Key (A, B, C)

Sol: (C) $P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(F \cap G) - P(G \cap E) + P(E \cap F \cap G)$

$= \dfrac{1}{8} + \dfrac{1}{6} + \dfrac{1}{4} - [P(E \cap F) + P(F \cap G) + P(E \cap G)] + \dfrac{1}{10} = \dfrac{13}{24} + \dfrac{1}{10} - \sum P(E \cap F)$

$\Rightarrow P(E \cup F \cup G) \leq \dfrac{13}{24}$

(C) is true

(D) $P(E^C \cap F^C \cap G^C) = 1 - P(E \cup F \cup G) \geq 1 - \frac{13}{24}$

$\Rightarrow P(E^C \cap F^C \cap G^C) \geq \frac{11}{24}$

(D) is true

(A) $P(E) = \frac{1}{8} \geq P(E \cap F \cap G^C) + P(E \cap F \cap G)$

$\Rightarrow \frac{1}{8} \geq P(E \cap F \cap G^C) + \frac{1}{10} \Rightarrow \frac{1}{8} - \frac{1}{10} \geq P(E \cap F \cap G^C)$

$\Rightarrow \frac{1}{40} \geq P(E \cap F \cap G^C)$ [(A) is Correct]

(B) $P(F) = \frac{1}{6} \geq P(E^C \cap F \cap G) + P(E \cap F \cap G)$

$\Rightarrow \frac{1}{6} - \frac{1}{10} \geq P(E^C \cap F \cap G)$

$\Rightarrow \frac{4}{60} \geq P(E^C \cap F \cap G)$

$\Rightarrow \frac{1}{15} \geq P(E^C \cap F \cap G)$

(B) is true

Q.14 For any 3×3 matrix M, let $|M|$ denote the determinant of M. Let I be the 3×3 identity matrix. Let E and F be two 3×3 matrices such that $(I - EF)$ is invertible. If $G = (I - EF)^{-1}$, then which of the following statements is (are) **TRUE** ?

(A) $|FE| = |I - FE||FGE|$ (B) $(I - FE)(I + FGE) = I$

(C) $EFG = GEF$ (D) $(I - FE)(I - FGE) = I$

Answer Key (A, B, C)

Sol: $G = (I - EF)^{-1} \Rightarrow G^{-1} = I - EF$

$\Rightarrow G(I - EF) = I = G - EFG \;\Theta\; G.G^{-1} = I = G^{-1}.G$

$\Rightarrow GEF = EFG$ **[**C is true]

(I –FE) (I + FGE) = I + FGE –FE –FEFGE

= I + FGE –FE –F (G –I) E

= I + FGE –FE –FGE + FE

= I [(B) is true]

(So (D) is False)

Given (I –FE) (I +FGE) = I ……. (1)

Now

FE (I + FGE)

= FE + FEFGE

= FE + F (G –I)E

= FE + FGE –FE

= FGE

$\Rightarrow$|FE| |I + FGE| = |FGE|

$\Rightarrow$|FE| × 1I – FE= |FGE| (from (1))

$\Rightarrow$|FE| = |I–FE| |FGE|

(Option (A) is true)

Q.15 For any positive integer n, let $S_n : (0, \infty) \to \mathbb{R}$ be defined by

$$S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1 + k(k+1)x^2}{x}\right),$$

where for any $x \in \mathbb{R}$, $\cot^{-1}(x) \in (0, \pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are) **TRUE** ?

(A) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all $x > 0$

(B) $\lim_{n\to\infty} \cot(S_n(x)) = x$, for all $x > 0$

(C) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$

(D) $\tan(S_n(x)) \le \frac{1}{2}$, for all $n \ge 1$ and $x > 0$

Answer Key (A, B)

Sol: $$S_n = \sum_{k=1}^{n} \tan^{-1}\left(\frac{x}{1+kx(kx+x)}\right)$$

$$= \sum_{k=1}^{n} \tan^{-1}\left(\frac{(kx+x)-(kx)}{1+(kx+x)(kx)}\right) = \sum_{k=1}^{n} \left[\tan^{-1}(kx+x) - \tan^{-1}(kx)\right]$$

$$S_n(x) = \tan^{-1}(nx+x) - \tan^{-1}x = \tan^{-1}\left(\frac{nx+x-x}{1+(n+1)x^2}\right) = \tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$$

$$\text{(A)}\ S_{10}(x) = \tan^{-1}\frac{10x}{1+11x^2} = \frac{\pi}{2} - \cot^{-1}\left(\frac{10x}{1+11x^2}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right) (x > 0)$$

$$\text{(B)}\ \lim_{n\to\infty} \cot(S_n(x)) = \lim_{n\to\infty} \frac{\frac{1}{n} + \left(1+\frac{1}{n}\right)x^2}{x} = x \quad (x > 0)$$

$$\text{(C)}\ S_3(x) = \tan^{-1}\frac{3x}{1+4x^2} = \frac{\pi}{4} \Rightarrow 4x^2 - 3x + 1 = 0 \Rightarrow x \notin R$$

$$\text{(D)}\ \tan(S_n(x)) = \frac{nx}{1+(n+1)x^2} : \forall n \ge 1; x > 0$$

Let $\frac{nx}{1+(n+1)x^2} \le \frac{1}{2} \forall n \ge 1; x > 0; n \in N$

$\Rightarrow (n+1)x^2 - 2nx + 1 \ge 0 \forall n \ge 1; x > 0; n \in N$

Discriminant of $y = (n+1)x^2 - 2nx + 1$ is

$\Delta = 4n^2 - 4(n+1)$ and $n \in N$

$\Delta < 0$ for n = 1; true for x > 0

$\Delta > 0$ for $n \ge 2 \Rightarrow$ for some x > 0

For which y < 0 as both roots of y = 0 will be positive.

$y = (n+1)x^2 - 2nx + 1, n \ge 2$

So, $y \ge 0 \forall n \ge 1; \forall x > 0; n \in N$ is false.

Q.16 For any complex number $w = c + id$, let $\arg(w) \in (-\pi, \pi]$, where $i = \sqrt{-1}$. Let α and β be real numbers such that for all complex numbers $z = x + iy$ satisfying $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$, the ordered pair (x, y) lies on the circle

$$x^2 + y^2 + 5x - 3y + 4 = 0$$

Then which of the following statements is (are) **TRUE** ?

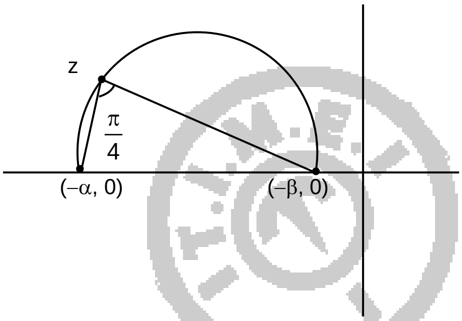
(A) $\alpha = -1$ (B) $\alpha\beta = 4$ (C) $\alpha\beta = -4$ (D) $\beta = 4$

Answer Key (B, D)

Sol: z lies on $x^2 + y^2 + 5x - 3y + 4 = 0$

put y =0;

$x^2 + 5x + 4 = 0 \Rightarrow x = -1; x = -4$



Now, $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4} \Rightarrow z + \alpha = (z+\beta)r\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

So, $z + \beta = z + 4 \Rightarrow \beta = 4$ and $z + \alpha = z + 1 \Rightarrow \alpha = 1$

SECTION 4

- This section contains **THREE** (03) questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks	:	+4	If ONLY the correct integer is entered.
Zero Marks	:	0	In all other cases.

Q.17 For $x \in \mathbb{R}$, the number of real roots of the equation

$$3x^2 - 4|x^2 - 1| + x - 1 = 0$$

is ___ .

Answer Key (4)

Sol: $3x^2 + x - 1 = 4|x^2 - 1|$

If $-1 \le x \le 1, 3x^2 + x - 1 = -4x^2 + 4 \Rightarrow 7x^2 + x - 5 = 0$

Say $f(x) = 7x^2 + x - 5$

$x = \frac{-1 \pm \sqrt{1+140}}{14} = \frac{-1 \pm \sqrt{141}}{14} \in [-1, 1]$

$\therefore$ two roots

If $x \in (-\infty,-1] \cup [1,\infty)$

$3x^2 + x - 1 = 4x^2 - 4 \Rightarrow x^2 - x - 3 = 0$

$\Rightarrow x = \dfrac{1 \pm \sqrt{1+12}}{2}$ which belong to the interval $(-\infty,-1] \cup [1,\infty)$

$\therefore$ Two roots

Hence total 4 roots

Q.18 In a triangle ABC, let $AB = \sqrt{23}, BC = 3$ and $CA = 4$. Then the value of

$$\frac{\cot A + \cot C}{\cot B}$$

is ___ .

Answer Key (2)

Sol:

$c = \sqrt{23}$; $a = 3$; $b = 4$

$$\frac{\frac{\cos A}{\sin A} + \frac{\cos C}{\sin C}}{\frac{\cos B}{\sin B}} = \frac{\sin(A+C)}{\sin A.\sin C} \times \frac{\sin B}{\cos B} = \frac{\sin^2 B}{\sin A.\sin C.\cos B}$$

$$= \frac{(2R\sin B)^2}{(2R\sin A)(2R\sin C).\cos B} = \frac{b^2}{ac\cos B} = \frac{b^2}{ac} \times \frac{2ac}{(a^2+c^2-b^2)}$$

$$= \frac{2b^2}{a^2+c^2-b^2} = \frac{2\times16}{9+23-16} = \frac{2\times16}{16} = 2$$

Q.19 Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in three-dimensional space, where $\vec{u}$ and $\vec{v}$ are unit vectors which are not perpendicular to each other and

$$\vec{u}\cdot\vec{w} = 1, \quad \vec{v}\cdot\vec{w} = 1, \quad \vec{w}\cdot\vec{w} = 4$$

If the volume of the parallelopiped, whose adjacent sides are represented by the vectors $\vec{u}, \vec{v}$ and $\vec{w}$, is $\sqrt{2}$, then the value of $|3\vec{u} + 5\vec{v}|$ is ___ .

Answer Key (7)

Sol: $\vec{w}.\vec{w} = |\vec{w}|^2 = 4 \Rightarrow |\vec{w}| = 2; [\vec{u}\,\vec{v}\,\vec{w}] = \sqrt{2} \Rightarrow |\vec{u}.(\vec{v}\times\vec{w})|^2 = \begin{vmatrix} \vec{u}^2 & \vec{u}.\vec{v} & \vec{u}.\vec{w} \\ \vec{v}.\vec{u} & \vec{v}^2 & \vec{v}.\vec{w} \\ \vec{w}.\vec{u} & \vec{w}.\vec{v} & \vec{w}^2 \end{vmatrix} = 2$

$$\Rightarrow \begin{vmatrix} 1 & \vec{u}.\vec{v} & 1 \\ \vec{v}.\vec{u} & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 2$$

$\Rightarrow \vec{u}.\vec{v} = \dfrac{1}{2}$

So, $|3\vec{u} + 5\vec{v}|^2 = 9|\vec{u}|^2 + 25|\vec{v}|^2 + 2.3\vec{u}.\vec{v}$

$\therefore |3\vec{u} + 5\vec{v}| = \sqrt{9 + 25 + 30\left(\dfrac{1}{2}\right)} = \sqrt{49} = 7$