QUESTIONS & KEYS FOR JEE (ADVANCED)-2021 (PAPER 2)

[MATHEMATICS]

SECTION 1

- This section contains **SIX (06)** question.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks	:	+4	If only (all) the correct option(s) is (are) chosen;
Partial Marks	:	+3	If all the four options are correct but ONLY three options are chosen;
Partial Marks	:	+2	If three or more options are correct but ONLY two options are chosen, both of which are correct.
Partial Marks	:	+1	If two or more options are correct but ONLY one option is chosen and it is a correction option.
Zero Marks	:	0	If unanswered;
Negative Marks	:	−2	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY options corresponding to correct answers, then
Choosing ONLY (A), (B) and (D) will get +4 marks;
Choosing ONLY (A) and (B) will get +2 marks;
Choosing ONLY (A) and (D) will get +2 marks;
Choosing ONLY (B) and (D) will get +2 marks;
Choosing ONLY (A) will get +1 mark;
Choosing ONLY (B) will get +1 mark;
Choosing ONLY (D) will get +1 mark;
Choosing no option(s) (i.e., the questions is unanswered) will get 0 marks and
choosing any other option(s) will get −2 marks.

Q.1 Let

$$S_1 = \{(i,j,k) : i,j,k \in \{1,2,\dots,10\}\},$$
$$S_2 = \{(i,j) : 1 \le i < j+2 \le 10,\ i,j \in \{1,2,\dots,10\}\},$$
$$S_3 = \{(i,j,k,l) : 1 \le i < j < k < l,\ i,j,k,l \in \{1,2,\dots,10\}\}$$

and

$$S_4 = \{(i,j,k,l) : i,j,k \text{ and } l \text{ are distinct elements in } \{1,2,\dots,10\}\}.$$

If the total number of elements in the set S_r is n_r, $r = 1,2,3,4$, then which of the following statements is (are) **TRUE** ?

(A) $n_1 = 1000$ (B) $n_2 = 44$ (C) $n_3 = 220$ (D) $\frac{n_4}{12} = 420$

Answer Key (A, B, D)

Sol: $n_1 = 10 \times 10 \times 10 = 1000$

For s_2

If i = 1, 2 then j = 1, 2, 3, ….8 $\Rightarrow$ $2 \times 8 = 16$ cases

If i = 3 then j = 2, 3, ….8 $\Rightarrow$ 7 cases

If i = 4 then j =3, 4, ….8 $\Rightarrow$ 6 cases

If i = 9 then j = 1 $\Rightarrow$ 1 case

$\therefore n_2 = (1+2+3+.........8)+8 = 44$

For s_3

$n_3 = {}^{10}C_4 = 210$

For s_4

$n_4 = {}^{10}P_4 = 12 \times {}^{10}C_4$

$\Rightarrow \frac{n_4}{12} = 420$

Q.2 Consider a triangle PQR having sides of lengths p, q and r opposite to the angles P, Q and R, respectively. Then which of the following statements is (are) **TRUE** ?

(A) $\cos P \geq 1 - \frac{p^2}{2qr}$

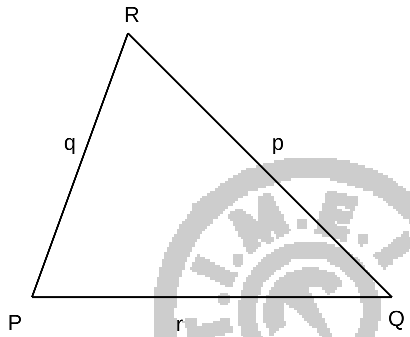
(B) $\cos R \geq \left(\frac{q-r}{p+q}\right)\cos P + \left(\frac{p-r}{p+q}\right)\cos Q$

(C) $\frac{q+r}{p} < 2\frac{\sqrt{\sin Q \sin R}}{\sin P}$

(D) If $p < q$ and $p < r$, then $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$

Answer Key (A, B)

Sol:



$$\cos p = \frac{q^2 + r^2 - p^2}{2qr} = \frac{q^2 + r^2}{2qr} - \frac{p^2}{2qr}$$

We know that $q^2 + r^2 \geq 2qr$

$\left[\because (q-r)^2 \geq 0 \Rightarrow q^2 + r^2 - 2qr \geq 0 \Rightarrow q^2 + r^2 \geq 2qr\right]$

$$\cos p = \frac{q^2 + r^2}{2qr} - \frac{p^2}{2qr} \geq 1 - \frac{p^2}{2qr}$$

$\therefore$ A is correct

Let us consider (B),

(B)$\Rightarrow (p+q)\cos R \geq (q-r)\cos P + (p-r)\cos Q$

$\Rightarrow (p\cos R + r\cos P) + (q\cos R + r\cos Q) \geq q\cos P + p\cos Q$

$\therefore p + q \geq r$

Which is always true.

$\therefore$(B) is correct

Let us consider (C),

$$\frac{q+r}{p} = \frac{k\sin Q + k\sin R}{k\sin P} \geq \frac{2\sqrt{\sin Q.\sin R}}{\sin P}$$

$\therefore$C is not correct

Let us consider (D),

$$\cos Q > \frac{p}{r} \Rightarrow \cos Q > \frac{k\sin P}{k\sin R}$$

$\Rightarrow \sin R\cos Q > \sin P$

$\Rightarrow \frac{1}{2}[\sin(R+Q)] + \sin[R-Q] > \sin P$

$\sin(180 - P) + \sin(R-Q) > 2\sin P$

$\Rightarrow \sin P + \sin(R-Q) > 2\sin P$

$\Rightarrow \sin(R-Q) > \sin P$

Which is not always true.

Q.3 Let $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}$ be a continuous function such that

$$f(0)=1 \text{ and } \int_0^{\frac{\pi}{3}} f(t)dt=0$$

Then which of the following statements is (are) **TRUE** ?

(A) The equation $f(x)-3\cos 3x=0$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

(B) The equation $f(x)-3\sin 3x=-\frac{6}{\pi}$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

(C) $\lim\limits_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}=-1$

(D) $\lim\limits_{x\to 0}\frac{\sin x\int_0^x f(t)dt}{x^2}=-1$

Answer Key (A, B, C)

Sol: Let g(x) = f(x) – 3cos3x

$$\int_0^{\pi/3} g(x)dx=\int_0^{\pi/3}(f(x)-3\cos 3x)dx$$

$$=\int_0^{\pi/3} f(x)dx-3\int_0^{\pi/3}\cos 3x dx$$

$$=0-\frac{3}{3}[\sin 3x]_0^{\pi/3}=0-0=0$$

$\therefore$ $y=g(x)$ crosses x axis at some point in the interval $\left(0,\frac{\pi}{3}\right)$

$\therefore$ $g(x)=0$ has a root in $\left(0,\frac{\pi}{3}\right)$

$\therefore$ A is correct

Let us consider (B)

Let $g(x)=f(x)-3\sin 3x+\frac{6}{\pi}$

$$\int_0^{\pi/3} g(x)dx=\int_0^{\pi/3}\left(f(x)-3\sin 3x+\frac{6}{\pi}\right)dx=\int_0^{\pi/3} f(x)dx-\int_0^{\pi/3}3\sin 3xdx+\frac{6}{\pi}\int_0^{\pi/3}1dx$$

$$=0+[\cos 3x]_0^{\pi/3}+\frac{6}{\pi}.\frac{\pi}{3}=-1-1+2=0$$

Let us consider (C)

$$\lim_{x\to 0}\frac{x\int_0^x f(t)dt}{\left(1-e^{x^2}\right)}=\lim_{x\to 0}\left[\frac{\left\{\int_0^x f(t)dt\right\}/x}{\left\{\left(1-e^{x^2}\right)/x^2\right\}}\right]=-1\times\lim_{x\to 0}\frac{f(x)}{1}=-1$$

$\therefore$C is correct

Let us consider (D)

$$\lim_{x\to 0}\left[\frac{\sin x.\int_0^x f(t)dt}{x^2}\right]=\lim_{x\to 0}\left[\frac{\sin x}{x}.\frac{\int_0^x f(t)dt}{x}\right]=1\times\lim_{x\to 0}\frac{f(x)}{1}=1$$

Q.4 For any real numbers α and β, let $y_{\alpha,\beta}(x)$, $x \in \mathbb{R}$, be the solution of the differential equation

$$\frac{dy}{dx} + \alpha y = xe^{\beta x}, \quad y(1) = 1.$$

Let $S = \{y_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{R}\}$. Then which of the following functions belong(s) to the set S ?

(A) $f(x) = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$

(B) $f(x) = -\frac{x^2}{2}e^{-x} + \left(e + \frac{1}{2}\right)e^{-x}$

(C) $f(x) = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right)e^{-x}$

(D) $f(x) = \frac{e^x}{2}\left(\frac{1}{2} - x\right) + \left(e + \frac{e^2}{4}\right)e^{-x}$

Answer Key (A, C)

Sol: $\text{I.F} = e^{\int \alpha dx} = e^{\alpha x}$

$\therefore ye^{\alpha x} = \int x.e^{\beta x}.e^{\alpha x} = \int xe^{(\alpha+\beta)x}dx$

Let $\alpha + \beta = 0$

$\Rightarrow ye^{\alpha x} = \int xe^{0x}dx = \int xdx = \frac{x^2}{2} + C$

$\Rightarrow ye^{\alpha x} = \frac{x^2}{2} + C$

Since $y(1) = 1, C = e^{\alpha} - \frac{1}{2}$

$\therefore ye^{\alpha x} = \frac{x^2 - 1}{2} + e^{\alpha}$

If $\alpha = 1$,

$y = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$

$\therefore$ A is correct

If $\alpha + \beta \neq 0$

$\Rightarrow ye^{\alpha x} = \frac{x.e^{(\alpha+\beta)x}}{\alpha+\beta} - \frac{1}{\alpha+\beta}\int e^{(\alpha+\beta)x}dx$

$\therefore ye^{\alpha x} = \frac{x.e^{(\alpha+\beta)x}}{\alpha+\beta} - \frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2}$

$\therefore C = e^{\alpha} - \frac{e^{\alpha+\beta}}{\alpha+\beta} - \frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}$

If $\alpha = \beta = 1$,

$y = \frac{(2x-1)}{4}e^x + \left(e - \frac{e^2}{2} + \frac{e^2}{4}\right)e^{-x}$

$\therefore y = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + e^{-x}\left(e - \frac{e^2}{4}\right)$

$\therefore$ C is correct

Q.5 Let O be the origin and $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA})$ for some $\lambda > 0$. If $|\overrightarrow{OB} \times \overrightarrow{OC}| = \frac{9}{2}$, then which of the following statements is (are) **TRUE** ?

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $-\frac{3}{2}$

(B) Area of the triangle OAB is $\frac{9}{2}$

(C) Area of the triangle ABC is $\frac{9}{2}$

(D) The acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{\pi}{3}$

Answer Key (A, B, C)

Sol: $\overrightarrow{OB} \times \overrightarrow{OC} = \frac{1}{2}\overrightarrow{OB} \times (\overrightarrow{OB} - \lambda\overrightarrow{OA}) = \frac{1}{2}\left[(\overrightarrow{OB} \times \overrightarrow{OB}) - \lambda(\overrightarrow{OB} \times \overrightarrow{OA})\right]$

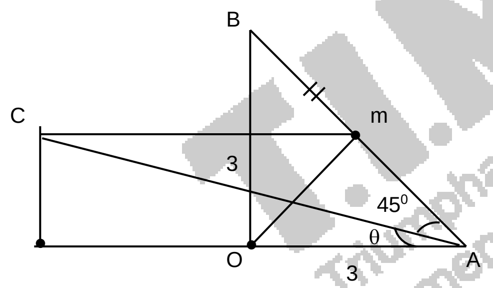
$= \frac{1}{2}\left[\vec{0} - \lambda(\overrightarrow{OB} \times \overrightarrow{OA})\right]$

$\therefore \overrightarrow{OB} \times \overrightarrow{OC} = \frac{\lambda}{2}(\overrightarrow{OB} \times \overrightarrow{OA})$

Taking modulus both sides,

$\frac{9\lambda}{2} = \frac{9}{2} \Rightarrow \lambda = 1$

$\overrightarrow{OC} = \frac{\overrightarrow{OB} - \overrightarrow{OA}}{2} = \frac{1}{2}.\overrightarrow{AB}$



Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA} = -\frac{3}{2}$

$\tan\theta = \frac{1}{3}$

Area of $\Delta ABC = \frac{9}{2}$

Acute angle between diagonals $= \tan^{-1}\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = \tan^{-1} 2$

Q.6 Let E denote the parabola $y^2 = 8x$. Let $P = (-2, 4)$, and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is (are) **TRUE** ?

(A) The triangle PFQ is a right-angled triangle

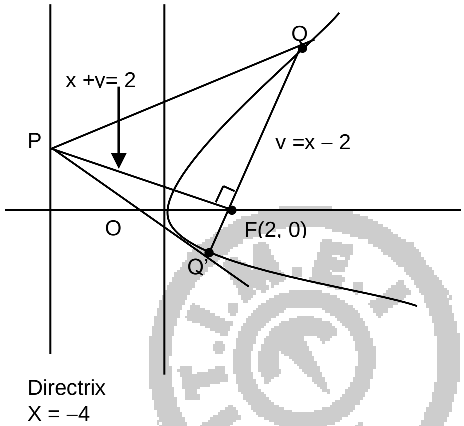
(B) The triangle QPQ' is a right-angled triangle

(C) The distance between P and F is $5\sqrt{2}$

(D) F lies on the line joining Q and Q'

Answer Key (A, B, D)

Sol:



SECTION 2

- This section contains **THREE (03)** questions stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate / round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks	:	+2	If ONLY the correct numerical value is entered at the designated place;
Zero Marks	:	0	In all other cases.

Question Stem for Question No. 7 and 8

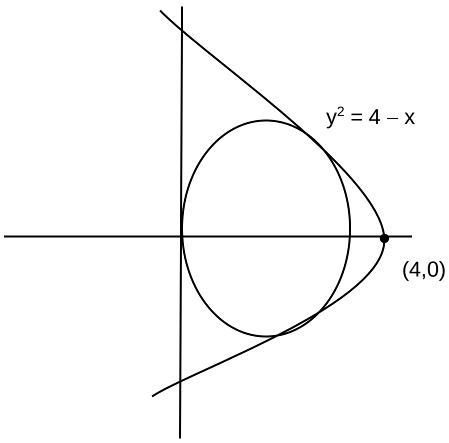
Question Stem

Consider the region $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0 \text{ and } y^2 \leq 4 - x\}$. Let $\mathcal{F}$ be the family of all circles that are contained in R and have centers on the x-axis. Let C be the circle that has largest radius among the circles in $\mathcal{F}$. Let (α, β) be a point where the circle C meets the curve $y^2 = 4 - x$.

Q.7 The radius of the circle C is ___ .

Answer Key (1.5)

Sol:



The circle passes through origin and centre lies on axis

$\therefore$ circle $\Rightarrow x^2 + y^2 + ax = 0$

Parabola $\Rightarrow y^2 = 4 - x$

Solving, $x^2 + x(a-1) + 4 = 0$

$b^2 - 4ac = 0 \Rightarrow (a-1)^2 - 16 = 0$

$\Rightarrow a = -3$

$\therefore x^2 + y^2 - 3x = 0 \Rightarrow r = \frac{3}{2} = 1.5$

Q.8 The value of α is ___ .

Answer Key (2.00)

Sol: $x^2 - 4x + 4 = 0 \Rightarrow x = 2$

$\therefore \alpha = 2$

Question Stem for Question No. 9 and 10

Question Stem

Let $f_1: (0,\infty) \to \mathbb{R}$ and $f_2: (0,\infty) \to \mathbb{R}$ be defined by

$$f_1(x) = \int_0^x \prod_{j=1}^{21} (t-j)^j \, dt, \qquad x > 0$$

and

$$f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450, \quad x > 0,$$

where, for any positive integer n and real numbers $a_1, a_2, \ldots, a_n$, $\prod_{i=1}^{n} a_i$ denotes the product of $a_1, a_2, \ldots, a_n$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, $i = 1, 2$, in the interval $(0, \infty)$.

Q.9 The value of $2m_1 + 3n_1 + m_1 n_1$ is ___ .

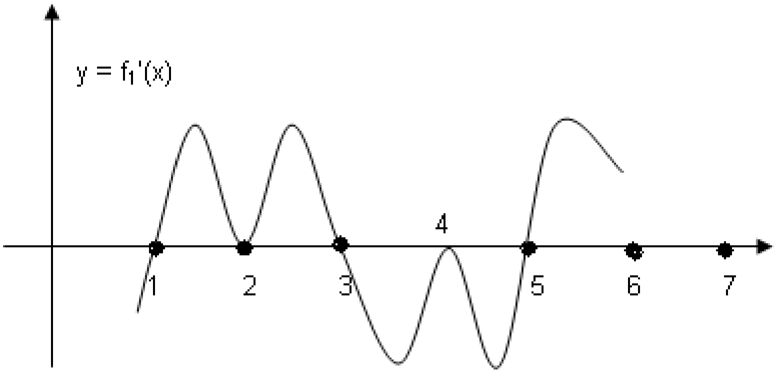
Answer Key (57.00)

Sol: $f_1(x) = \int_0^x \prod_{j=1}^{21} (t-j)^j dt$

$$f_1'(x) = \frac{d}{dx}\int_0^x \prod_{j=1}^{21}(t-j)^j\,dt$$

$$\prod_{j=1}^{21}(x-j)^j$$

$$= (x-1)(x-2)^2(x-3)^3........(x-21)^{21}$$



Clearly $m_1 = 6, n_1 = 5$

$\therefore 2m_1 + 3n_1 + m_1 n_1 = 57$

Q.10 The value of $6m_2 + 4n_2 + 8m_2 n_2$ is ___.

Answer Key (6.00)

Sol: $f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450$

$f_2'(x) = (50)(98)(x-1)^{49} - (600)(49)(x-1)^{48}$

$= 50 \times 2 \times 49 \times (x-1)^{48}(x-7)$ (after arrangement)

$f_2'(7^+) > 0$ and $f_2'(7^-) < 0$

$\therefore$ point of minima $= 7$

$\therefore m_2 = 1$

No point of maxima $\Rightarrow n_2 = 0$

$\therefore 6m_2 + 4n_2 + 8m_2 n_2 = 6$

Question Stem for Question No. 11and 12

Question Stem

Let $g_i : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to \mathbb{R}, i = 1, 2$, and $f : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to \mathbb{R}$ be functions such that

$$g_1(x) = 1,\ g_2(x) = |4x - \pi| \text{ and } f(x) = \sin^2 x, \text{ for all } x \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$$

Define

$$S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x) \cdot g_i(x)\,dx, \quad i = 1, 2$$

Q.11 The value of $\frac{16S_1}{\pi}$ is ____ .

Answer Key (2.00)

Sol: $S_1 = \int_{\pi/8}^{3\pi/8} f(x)dx = \int_{\pi/8}^{3\pi/8} \sin^2 x dx = \int_{\pi/8}^{3\pi/8} \sin^2\left(\frac{\pi}{8}+\frac{3\pi}{8}-x\right)dx = \int_{\pi/8}^{3\pi/8} \cos^2 x dx$

$2S_1 = \int_{\pi/8}^{3\pi/8} \left(\sin^2 x + \cos^2 x\right)dx = [x]_{\pi/8}^{3\pi/8} = \frac{2\pi}{8} = \frac{\pi}{4}$

$\therefore \frac{16S_1}{\pi} = 2$

Q.12 The value of $\frac{48S_2}{\pi^2}$ is ____.

Answer Key (1.5)

Sol: $S_2 = \int_{\pi/8}^{3\pi/8} \sin^2 x.|4x-\pi|dx = \int_{\pi/8}^{3\pi/8} \sin^2\left(\frac{\pi}{8}+\frac{3\pi}{8}-x\right).\left|4\left(\frac{\pi}{8}+\frac{3\pi}{8}-x\right)-\pi\right|dx$

$= \int_{\pi/8}^{3\pi/8} \cos^2 x.|\pi-4x|dx$

$2S_2 = \int_{\pi/8}^{3\pi/8} |4x-\pi|\left(\sin^2 x+\cos^2 x\right)dx = \int_{\pi/8}^{3\pi/8} |4x-\pi|dx = \int_{\pi/8}^{\pi/4} (\pi-4x)dx + \int_{\pi/4}^{3\pi/8} (4x-\pi)dx = \frac{\pi^2}{16}$

$\therefore \frac{48S_2}{\pi^2} = \frac{3}{2} = 1.5$

SECTION 3

- This section contains **TWO (02) paragraphs** Based on each paragraph, there are **TWO (02)** questions.
- Each questions has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks	:	+3	If ONLY the correct option is chosen;
Zero Marks	:	0	If none of the options is chosen (i.e., the question is unanswered);
Negative Marks	:	−1	In all other cases.

Paragraph

Let

$$M = \{(x,y) \in \mathbb{R}\times\mathbb{R} : x^2+y^2 \le r^2\},$$

where $r > 0$. Consider the geometric progression $a_n = \frac{1}{2^{n-1}}, n = 1,2,3,\ldots$. Let $S_0 = 0$ and, for $n \ge 1$, let S_n denote the sum of the first n terms of this progression. For $n \ge 1$, let C_n denote the circle with center $(S_{n-1}, 0)$ and radius a_n, and D_n denote the circle with center (S_{n-1}, S_{n-1}) and radius a_n.

Q.13 Consider M with $r = \frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M. Let l be the maximum possible number of circles among these k circles such that no two circles intersect. Then

(A) $k + 2l = 22$ (B) $2k + l = 26$ (C) $2k + 3l = 34$ (D) $3k + 2l = 40$

Answer Key (D)

Sol: $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \text{....to 'n' terms} = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 2 - \frac{1}{2^{n-1}}$

Centre $C_n : \left(2 - \frac{1}{2^{n-2}}, 0\right)$ and $r = \frac{1}{2^{n-1}}$

Given $r = \frac{1025}{513} < 2$

$2 - \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} < \frac{1025}{513}$

$\therefore k = 10, l = 5$

$3k + 2l = 40$

Q.14 Consider M with $r = \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is

(A) 198 (B) 199 (C) 200 (D) 201

Answer Key (B)

Sol: $\sqrt{2}S_{n-1} < \frac{(2^{199}-1)}{2^{198}}.\sqrt{2}$

$\Rightarrow \frac{\sqrt{2}}{2^{n-2}} > \frac{\sqrt{2}}{2^{198}} + \frac{1}{2^{n-1}}$

$\therefore n = 199$

Paragraph

Let $\psi_1 : [0,\infty) \to \mathbb{R}$, $\psi_2 : [0,\infty) \to \mathbb{R}$, $f : [0,\infty) \to \mathbb{R}$ and $g : [0,\infty) \to \mathbb{R}$ be functions such that $f(0) = g(0) = 0$,

$$\psi_1(x) = e^{-x} + x, \quad x \geq 0,$$

$$\psi_2(x) = x^2 - 2x - 2e^{-x} + 2, \quad x \geq 0,$$

$$f(x) = \int_{-x}^{x} (|t| - t^2)e^{-t^2}\, dt, \quad x > 0$$

and

$$g(x) = \int_0^{x^2} \sqrt{t}\, e^{-t}\, dt, \quad x > 0.$$

Q.15 Which of the following statements is **TRUE** ?

(A) $f(\sqrt{\ln 3}) + g(\sqrt{\ln 3}) = \frac{1}{3}$

(B) For every $x > 1$, there exists an $\alpha \in (1, x)$ such that $\psi_1(x) = 1 + \alpha x$

(C) For every $x > 0$, there exists a $\beta \in (0, x)$ such that $\psi_2(x) = 2x(\psi_1(\beta) - 1)$

(D) f is an increasing function on the interval $\left[0, \frac{3}{2}\right]$

Answer Key (C)

Sol: $f'(x) = e^{-x^2}(|x| - x^2) + (|x| - x^2)e^{-x^2} = 2(x - x^2)e^{-x^2}$

$g'(x) = 2x^2e^{-x^2}$

$f'(x) + g'(x) = 2xe^{-x^2}$

$\Rightarrow \int [f'(x) + g'(x)]dx = \int 2xe^{-x^2}dx$

$\Rightarrow f(x) + g(x) = -e^{-x^2} + C$

$f(x) + g(x) = -e^{-x^2} + 1$

$\Rightarrow f(\sqrt{\ln 3}) + g(\sqrt{\ln 3}) = -e^{-\ln 3} + 1 = 1 - \frac{1}{3} = \frac{2}{3}$

A is not correct

Let h(x) = $\psi_1(x) - 1 - \alpha x$

$= e^{-x} + x - (1 + \alpha x)$

$h(1) = e^{-1} + 1 - 1 - \alpha < 0$

$h'(x) = -e^{-x} - \alpha + 1 > 0$

$\therefore$ h(x) is strictly increasing

$\therefore$ B is not correct

Let $\psi_2(x) = 2(\psi_1(B) - 1)$

Applying mean value theorem in [0, x]

$\psi_2'(\beta) = \frac{\psi_2(x) - \psi_2(0)}{x}$

$\Rightarrow 2\beta - 2 + 2e^{-\beta} = \frac{\psi_2(x) - \psi_2(0)}{x}$

$\therefore \psi_2(x) = 2x.\psi_1(\beta - 1)$ has one solution

(C) is correct

Q.16 Which of the following statements is **TRUE** ?

(A) $\psi_1(x) \leq 1$, for all $x > 0$

(B) $\psi_2(x) \leq 0$, for all $x > 0$

(C) $f(x) \geq 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

(D) $g(x) \leq \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$, for all $x \in \left(0, \frac{1}{2}\right)$

Answer Key (D)

Sol: Let $f(x) = g(x) - \frac{2}{3}x^3 + \frac{2}{5}x^5 - \frac{1}{7}x^7$, $x \in \left(0, \frac{1}{2}\right)$

$$f'(x) = 2x^2\left(e^{-x^2}\right) - 2x^2 + 2x^4 - x^6 = 2x^2\left[1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + ...\right] - 2x^2 + 2x^4 - x^6$$

$$= -\frac{x^8}{3} + \frac{x^{10}}{12} - ....$$

$\therefore f'(x) \le 0$

$\Rightarrow f(x)$ is strictly decreasing

$\therefore f(x) \le 0$

SECTION 4

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks	:	+4	If ONLY the correct integer is entered.
Zero Marks	:	0	In all other cases.

Q.17 A number is chosen at random from the set $\{1, 2, 3, \ldots, 2000\}$. Let p be the probability that the chosen number is a multiple of 3 or a multiple of 7. Then the value of $500p$ is ___ .

Answer Key (214)

Sol: A = {3, 6, 9, ….. 1998}

$$n(A) = \frac{1998 - 3}{3} + 1 = 666$$

B = {7, 14, 21, …… 1995}

$$n(B) = \frac{1995 - 7}{7} + 1 = 285$$

$A \cap B = \{21, 42, ...1995\}$

$\Rightarrow n(A \cap B) = 95$

$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B) = 666 + 285 - 95 = 856$

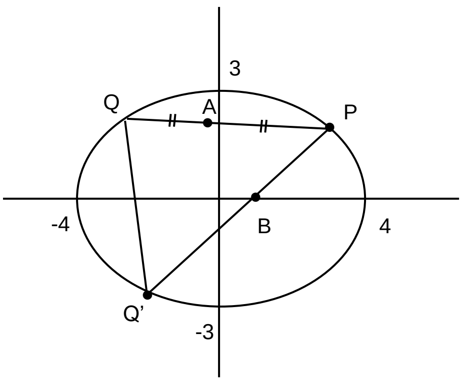
$$P = \frac{856}{2000} = 0.428 \text{ P}$$

$\Rightarrow 500P = 214$

Q.18 Let E be the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. For any three distinct points P, Q and Q' on E, let $M(P, Q)$ be the mid-point of the line segment joining P and Q, and $M(P, Q')$ be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between $M(P, Q)$ and $M(P, Q')$, as P, Q and Q' vary on E, is ___ .

Answer Key (4)

Sol:



Midpoint of PQ = A
Midpoint of PQ' = B

$AB = \frac{1}{2} QQ'$

$AB_{max} = \frac{8}{2} = 4$

Q.19 For any real number x, let $[x]$ denote the largest integer less than or equal to x. If

$$I = \int_0^{10} \left[\sqrt{\frac{10x}{x+1}} \right] dx ,$$

then the value of $9I$ is __ .

Answer Key (182)

Sol: $\frac{d}{dx}\left[\frac{10x}{x+1}\right] = \frac{10}{(x+1)^2} > 0 \ \forall \ x \in [0,10]$

$\therefore \frac{10x}{x+1}$ is an increasing function

$0 \le f(x) \le \sqrt{\frac{100}{11}}$

$$I = \int_0^{1/9} f(x)dx + \int_{1/9}^{2/3} f(x)dx + \int_{2/3}^{9} f(x)dx + \int_9^{10} f(x)dx = 0 + \int_{1/9}^{2/3} 1dx + \int_{2/3}^{9} 2dx + \int_9^{10} 3\,dx = \frac{182}{9}$$

$\therefore 9I = 182$