Vector Integral Calculus

Integration

Line Integral: Let $\bar{F}(x, y, z)$ be a vector function defined on a region of space and let C be curve in that region, the integral $\int_c \bar{F} \cdot d\bar{r}$ is called the line integral.

For Riemann Integration,

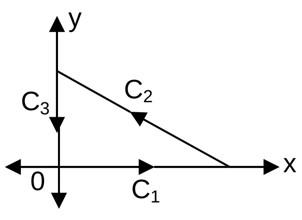
$\int_{x=a}^{x=b} f\,dx$ the limits of integration are along the line segment joining (a, 0), (b, 0), where a < b.

Here instead of line, we integrate along the curve C.

Circulation: The line integral around a closed curve C denoted by $\oint_c F.dr$ is called circulation of **F** around C.

Eg 1. Evaluate **F**·dr, where **F** = $xy\,\hat{i} + y^2\,\hat{j}$ along the triangle x = 0, y = 0 and x + y = 1 in the first quadrant.

Sol.



$$\int_c F \cdot dr = \int_{C_1} xy\,dx + y^2 dy + \int_{C_2} xy\,dx + y^2 dy + \int_{C_3} xy\,dx + y^2 dy$$

C_1	C_2	C_3
y = 0	y = 1 – x	x = 0
0 < x < 1	0 < x < 1	dx = 0
dy = 0	dy = – dx	1 < y < 0

$$= \int_{x=0}^{1} [x(0)dx + 0 + \int_{x=0}^{1} x(1-x)dx + (1-x)^2(-dx) + \int_{1}^{0} y^2 dy$$

$$= \int_0^1 (x - x^2 - 1 - x^2 + 2x)\,dx + \int_1^0 y^2 dy$$

$= \int_0^1 (-2x^2 + 3x - 1)\, dx - \int_0^1 y^2\, dy$

$= (-2/3 + 3/2 - 1 - 1/3) = -1/2.$

Surface Integral: Let S be a closed surface, then the normal surface integral $\int_s \mathbf{F}\ \mathbf{N}\ ds$ is called the flux of F over S.

Cartesian Form: Let $F(r) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ where, F_1, F_2, F_3 are continuous and differentiable functions of x, y, z. If cosα, cosβ and cosγ be the direction cosines of the unit normal **N**, then

$\mathbf{N} = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma.$

$\therefore \int_S F \,.\, N\, ds = \int_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, ds$

But then ds cosα, ds cosβ and ds cosγ are the projections of ds on yz, zx and xy planes. If dx, dy, dz are the differentials along the areas then

ds cosα=dy dz; ds cosβ = dz dx; ds cosγ = dx dy.

$\therefore \int_S F \cdot N\, ds = \iint_S (F_1 dy\, dz + F_2\, dz\, dx + F_3\, dx dy)$

Note: If R_1 is the projection of S on xy-plane, then

$\int_S \mathbf{F \cdot N} ds = \iint_{R1} F \cdot N \frac{dx\ dy}{\cos\gamma}$

$= \mathbf{F \cdot N.} \frac{dx\, dy}{|N.\hat{k}|} \quad (|N \cdot \hat{k}| = \cos\gamma)$

Equivalently, $\int_S F \cdot N ds = \int\int_{R2} F \cdot N \frac{dy\ dz}{|N \cdot \hat{i}|} = \int\int_{R3} F \cdot N \frac{dz\ dx}{|N \cdot \hat{j}|}$

Volume Integral

$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z)\, dz\, dy\, dx$

$$= \int_{x_1}^{x_2} \left[\int_{y_1}^{y_2} \left[\int_{z_1}^{z_2} f(x, y, z)\, dz \right] dy \right] dx$$